Theory and Applications of Dynamical Systems

Part II.
Pattern Generation Problems

Song-Sun Lin
Department of Applied Mathematics
National Chiao-Tung University
Hsin-Chu, Taiwan

Contents

Chapter 1

2-Dimensional Patterns Generation Problems

§ 1.1 Introduction

Many systems have been studied as models for spatial pattern formation in biology, chemistry, engineering and physics. Lattices play important roles in modeling underlying spatial structures. Notable examples include models arising from biology[7, 8, 21, 22, 23, 33, 34, 35], chemical reaction and phase transitions [4, 5, 11, 12, 13, 14, 24, 41, 43], image processing and pattern recognition [11, 12, 15, 16, 17, 18, 19, 25, 40], as well as materials science[9, 20, 26]. Stationary patterns play a critical role in investigating of the long time behavior of related dynamical systems. In general, multiple stationary patterns may induce complicated phenomena of such systems.

In Lattice Dynamical Systems(LDS), especially Cellular Neural Networks (CNN), the set of global stationary solutions (global patterns) has received considerable attention in recent years (e.g.[1, 2, 6, 10, 27, 28, 29, 30, 31, 32, 36, 37]). When the mutual interaction between states of a system is local, the state at each lattice point is influenced only by its finitely many neighborhood states. The admissible (or allowable) local patterns are introduced and defined on a certain finite lattice. The admissible global patterns on the entire lattice space are then glued together from those admissible local patterns. More precisely, let $\mathcal{S}$ be a finite set of p elements (symbols, colors or letters of an alphabet). Where $\mathbf{Z}^d$ denotes the integer lattice on $\mathbf{R}^d$, and $d \geq 1$ is a positive integer representing the lattice dimension. Then, function $U : \mathbf{Z}^d \rightarrow \mathcal{S}$ is called a global pattern. For each $\alpha \in \mathbf{Z}^d$, we write $U(\alpha)$ as u_α. The set of all patterns $U : \mathbf{Z}^d \rightarrow \mathcal{S}$ is denoted by

$$\Sigma_p^d \equiv \mathcal{S}^{\mathbf{Z}^d},$$

i.e., Σ_p^d is the set of all patterns with p different colors in d-dimensional lattice. As for local patterns, i.e., functions defined on (finite) sublattices, for a given d-tuple $N = (N_1, N_2, \cdots, N_d)$ of positive integers, let

$$\mathbf{Z}_N = \{(\alpha_1, \alpha_2, \cdots, \alpha_d) : 1 \leq \alpha_k \leq N_k, 1 \leq k \leq d\}$$

be an $N_1 \times N_2 \times \cdots N_d$ finite rectangular lattice. Denoted by $\widetilde{N} \geq N$ if $\widetilde{N}_k \geq N_k$ for all $1 \leq k \leq d$. The set of all local patterns defined on $\mathbf{Z}_N$ is denoted by

$$\Sigma_N \equiv \Sigma_{N,p} \equiv \{U|_{\mathbf{Z}_N} : U \in \Sigma_p^d\}.$$

Under many circumstances, only a(proper) subset $\mathcal{B}$ of Σ_N is admissible (allowable or feasible). In this case, local patterns in $\mathcal{B}$ are called basic patterns and $\mathcal{B}$ is called the basic set. In a one dimensional case, $\mathcal{S}$ consists of letters of an alphabet, and $\mathcal{B}$ is also called a set of allowable words of length N.

Consider a fixed finite lattice $\mathbf{Z}_N$ and a given basic set $\mathcal{B} \subset \Sigma_N$. For larger finite lattice $\mathbf{Z}_{\widetilde{N}} \supset \mathbf{Z}_N$, the set of all local patterns on $\mathbf{Z}_{\widetilde{N}}$ which can be generated by $\mathcal{B}$ is denoted as $\Sigma_{\widetilde{N}}(\mathcal{B})$. Indeed, $\Sigma_{\widetilde{N}}(\mathcal{B})$ can be characterized by

$$\begin{aligned} \Sigma_{\widetilde{N}}(\mathcal{B}) \;\; = \{ \;\; & U \in \Sigma_{\widetilde{N}} : U_{\alpha+N} = V_N \; for \; any \; \alpha \in \mathbf{Z}^d \; with \; \mathbf{Z}_{\alpha+N} \subset \mathbf{Z}_{\widetilde{N}} \\ & and \; some \; V_N \in \mathcal{B}\}, \end{aligned}$$

where

$$\alpha + N = \{(\alpha_1 + \beta_1, \cdots, \alpha_d + \beta_d) : (\beta_1, \cdots, \beta_d) \in N\},$$

and

$$U_{\alpha+N} = V_N \; means \; u_{\alpha+\beta} = v_\beta \; for \; each \; \beta \in \mathbf{Z}_N.$$

Similarly, the set of all global patterns which can be generated by $\mathcal{B}$ is denoted by

$$\Sigma(\mathcal{B}) = \{U \in \Sigma_p^d : U_{\alpha+N} = V_N \; for \; any \; \alpha \in \mathbf{Z}^d \; with \; some \; V_N \in \mathcal{B}\}.$$

The following questions arise :

(1) Can we find a systematic means of constructing $\Sigma_{\widetilde{N}}(\mathcal{B})$ from $\mathcal{B}$ for $\mathbf{Z}_{\widetilde{N}} \supset \mathbf{Z}_N$?

(2) What is the complexity (or spatial entropy) of $\{\sum_{\widetilde{N}}(\mathcal{B})\}_{\widetilde{N} \geq N}$?

The spatial entropy $h(\mathcal{B})$ of $\Sigma(\mathcal{B})$ is defined as follows :
Let

$$\Gamma_{\widetilde{N}}(\mathcal{B}) = card(\Sigma_{\widetilde{N}}(\mathcal{B})), \tag{1.1.1}$$

the number of distinct patterns in $\Sigma_{\widetilde{N}}(\mathcal{B})$. The spatial entropy $h(\mathcal{B})$ is defined as

$$h(\mathcal{B}) = \lim_{\widetilde{N} \to \infty} \frac{1}{\widetilde{N_1} \cdots \widetilde{N_d}} \log \Gamma_{\widetilde{N}}(\mathcal{B}), \tag{1.1.2}$$

where $\widetilde{N} = (\widetilde{N_1}, \widetilde{N_2}, ..., \widetilde{N_d})$ be a d-tuple positive integers, which is well-defined and exists (e.g. [13]). The spatial entropy, which is an analogue to topological entropy in dynamical system, has been used to measure a kind of complexity in LDS (e.g. [13], [42]).

In a one dimensional case, the above two questions can be answered by using transition matrix. Indeed, for a given basic set $\mathcal{B}$, we can associate the transition matrix $\mathbf{T}(\mathcal{B})$ to $\mathcal{B}$. Then the spatial entropy $h(\mathcal{B}) = \log \lambda$, where λ is the largest eigenvalue of $\mathbf{T}(\mathcal{B})$ (e.g. [29, 41]). On the other hand, for higher dimensional cases, constructing $\Sigma_{\widetilde{N}}(\mathcal{B})$ systematically and computing $\Gamma_{\widetilde{N}}(\mathcal{B})$ effectively for a large $\widetilde{N}$ are extremely difficult.

In the two dimensional case, Chow et al. [13] estimated lower bounds of the spatial entropy for some problems in LDS. Later, using a "building block" technique, Juang and Lin [29] studied the patterns generation and obtained lower bounds of the spatial entropy for CNN with square-cross or diagonal-cross templates. For CNN with general templates, Hsu et al [27] investigated the generation of admissible local patterns and obtained the basic set for any parameter, i.e., the first step in studying the patterns generation problem. Meanwhile, given a set of symbols $\mathcal{S}$ and a pair consisting of a horizontal transition matrix H and a vertical transition matrix V, Juang et al [30] defined m-th order transition matrices $T^{(m)}_{H,V}$ and $\bar{T}^{(m)}_{H,V}$ for each $m \geq 1$ and, in doing so, obtained the recursion formulas for both $T^{(m)}_{H,V}$ and $\bar{T}^{(m)}_{H,V}$. Furthermore, they proved that $T^{(m)}_{H,V}$ and $\bar{T}^{(m)}_{H,V}$ have the same maximum eigenvalue λ_m and spatial entropy $h(H,V) = \lim\limits_{m\to\infty} \frac{\log \lambda_m}{m}$. For a certain class of H,V, the recursion formulas for $T^{(m)}_{H,V}$ and $\bar{T}^{(m)}_{H,V}$ yield recursion formulas for λ_m explicitly and the exact entropy. On the other hand, for the patterns generation problem Lin and Yang [37] worked on the 3-cell L-shaped lattice, i.e., N= . They developed an algorithm to investigate how patterns are generated on larger lattices from smaller one. Their algorithm treated all patterns in $\Sigma_{\widetilde{N}}(\mathcal{B})$ as entries and arranged them in a "counting matrix" $M_{\widetilde{N}}(\mathcal{B})$. A good arrangement of $M_{\widetilde{N}}(\mathcal{B})$ implies an easier extension to $M_{\widetilde{\widetilde{N}}}(\mathcal{B})$ for a larger lattice $\widetilde{\widetilde{N}} \supset \widetilde{N}$ and effective counting of the number of elements in $\Sigma_{\widetilde{N}}(\mathcal{B})$. Upper and lower bounds of spatial entropy were also obtained. Next, there are some relations with matrix shift [13], that details will appear in section 1.3.4.

Motivated by the counting matrix $M_N(\mathcal{B})$ of [37] and the recursion formulas for transition matrices in [30], this work introduces the "ordering matrix" $\mathbf{X}_2$ for $\Sigma_{2\ell\times 2\ell}$ to study the patterns generation and obtain recursion formulas for $\mathbf{X}_n$ for $\Sigma_{2\ell\times n\ell}$ where $\ell \geq 1$ is a fixed positive integer and $n \geq 2$. The recursion formulas for $\mathbf{X}_n$ imply the recursion formula for the associated transition matrices $\mathbf{T}_n(\mathcal{B})$ of $\Sigma_{2\ell\times n\ell}(\mathcal{B})$, i.e., a generalization of the recursion formulas in [30]. Notably, a different ordering matrix $\widetilde{\mathbf{X}}_2$ for $\Sigma_{2\ell\times 2\ell}$ induces different recursion formulas of $\widetilde{\mathbf{X}}_n$ for $\Sigma_{2\ell\times n\ell}$ and $\widetilde{\mathbf{T}_n}(\mathcal{B})$. Among them, $\mathbf{X}_2$ defined in (1.2.9) yields a simple recursion formula (1.3.16) and rewriting rule (1.3.14), which enabling us to compute the maximum eigenvalue of $\mathbf{T}_n$ effectively. The computations or estimates of λ_n are interesting problems in linear algebra and numerical linear algebra. Owing to the similarity property of (1.3.16) or (1.3.14) of transition matrices $\{\mathbf{T}_n\}_{n=2}^{\infty}$, we show that for a certain class of $\mathcal{B}$, λ_n satisfies certain recursion relations and $h(\mathcal{B})$ can be computed explicitly.

In $d \geq 3$, the structure of ordering matrix and transition matrices has been explored, and it can be found in [3].

The rest of this paper is organized as follows. Section 1.2 describes a two dimensional case by thoroughly investigating $\Sigma_{2\times 2}$ and introducing the ordering matrix $\mathbf{X}_2$ of patterns in $\Sigma_{2\times 2}$. The ordering matrix $\mathbf{X}_n$ on $\Sigma_{2\times n}$ is then constructed from $\mathbf{X}_2$ recursively. Finally, section 1.3 derives higher order transition matrices $\mathbf{T}_n$ from $\mathbf{T}_2$ and computes λ_n explicitly for a certain type of $\mathbf{T}_2$.

§ 1.2 Two Dimensional Patterns

This section describes two dimensional patterns generation. For clarity, we begin by the studying two symbols, i.e., $\mathcal{S} = \{0, 1\}$. On a fixed finite lattice $\mathbf{Z}_{m_1\times m_2}$, we first give a ordering $\chi = \chi_{m_1\times m_2}$ on $\mathbf{Z}_{m_1\times m_2}$ by

$$\chi((\alpha_1, \alpha_2)) = m_2(\alpha_1 - 1) + \alpha_2\ , \tag{1.2.1}$$

i.e.,

(1.2.2)

m_2	$2m_2$		$m_1 m_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	$m_2 + 1$		$(m_1 - 1)m_2 + 1$

The ordering χ of (1.2.1) on $\mathbf{Z}_{m_1\times m_2}$ can now be passed to $\Sigma_{m_1\times m_2}$. Indeed, for each $U = (u_{\alpha_1,\alpha_2}) \in \Sigma_{m_1\times m_2}$, define

$$\begin{aligned} \chi(U) &\equiv \chi_{m_1\times m_2}(U) \\ &= 1 + \sum_{\alpha_1=1}^{m_1}\sum_{\alpha_2=1}^{m_2} u_{\alpha_1\alpha_2} 2^{m_2(m_1-\alpha_1)+(m_2-\alpha_2)}. \end{aligned} \tag{1.2.3}$$

Obviously, there is an one-to-one correspondence between local patterns in $\Sigma_{m_1\times m_2}$ and positive integers in the set $\mathbf{N}_{2^{m_1 m_2}} = \{k \in \mathbf{N} : 1 \leq k \leq$

$2^{m_1,m_2}\}$, where $\mathbf{N}$ is the set of positive integers. Therefore, U is referred to herein as the $\chi(U)$-th element in $\Sigma_{m_1\times m_2}$. By identifying the pictorial patterns by numbers $\chi(U)$, it becomes highly effective in proving theorems since computations can now be performed on $\chi(U)$. In a two dimensional case, we will keep the ordering (1.2.1)$\sim$ (1.2.3) χ on $\mathbf{Z}_{m_1\times m_2}$ and $\Sigma_{m_1\times m_2}$, respectively.

1.2.1 Ordering Matrices

For $1\times n$ pattern $U=(u_k), 1\leq k\leq n$ in $\Sigma_{1\times n}$, as in (1.2.3), U is assigned the number

$$i=\chi(U)=1+\sum_{k=1}^{n}u_k 2^{(n-k)}. \tag{1.2.4}$$

As denoted by the $1\times n$ column pattern $x_{n;i}$,

$$x_{n;i}=\begin{bmatrix} u_n\\ \vdots\\ u_1\end{bmatrix} \quad or \quad \begin{array}{|c|}\hline u_n\\ \hline \vdots\\ \hline u_1\\ \hline\end{array}\;. \tag{1.2.5}$$

In particular, when $n=2$, as denoted by $x_i=x_{2;i}$,

$$i=1+2u_1+u_2$$

and

$$x_i=\begin{bmatrix} u_2\\ u_1\end{bmatrix} \quad or \quad \begin{array}{|c|}\hline u_2\\ \hline u_1\\ \hline\end{array}\;. \tag{1.2.6}$$

A 2×2 pattern $U=(u_{\alpha_1\alpha_2})$ can now be obtained by a horizontal direct sum of two 1×2 patterns, i.e.,

$$\begin{aligned} x_{i_1i_2} &\equiv x_{i_1}\oplus x_{i_2}\\ &\equiv \begin{bmatrix} u_{12} & u_{22}\\ u_{11} & u_{21}\end{bmatrix} \quad or \quad \begin{array}{|c|c|}\hline u_{12} & u_{22}\\ \hline u_{11} & u_{21}\\ \hline\end{array}\;, \end{aligned} \tag{1.2.7}$$

where

$$i_k=1+2u_{k1}+u_{k2}, \qquad 1\leq k\leq 2. \tag{1.2.8}$$

Therefore, the complete set of all $16(=2^{2\times 2})$ 2×2 patterns in $\Sigma_{2\times 2}$ can be listed by a 4×4 matrix $\mathbf{X}_2=[x_{i_1 i_2}]$ with 2×2 pattern $x_{i_1 i_2}$ as its entries in

$$
(1.2.9)\qquad
\begin{array}{c}
 \\
\begin{array}{|c|}\hline 0\\\hline 0\\\hline\end{array}\\[6pt]
\begin{array}{|c|}\hline 1\\\hline 0\\\hline\end{array}\\[6pt]
\begin{array}{|c|}\hline 0\\\hline 1\\\hline\end{array}\\[6pt]
\begin{array}{|c|}\hline 1\\\hline 1\\\hline\end{array}
\end{array}
\begin{array}{c}
\begin{array}{cccc}
\begin{array}{|c|}\hline 0\\\hline 0\\\hline\end{array} & \begin{array}{|c|}\hline 1\\\hline 0\\\hline\end{array} & \begin{array}{|c|}\hline 0\\\hline 1\\\hline\end{array} & \begin{array}{|c|}\hline 1\\\hline 1\\\hline\end{array}
\end{array}\\
\left(\begin{array}{cccc}
\begin{array}{|c|c|}\hline 0&0\\\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&0\\\hline 0&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 0&1\\\hline\end{array}\\[6pt]
\begin{array}{|c|c|}\hline 1&0\\\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&0\\\hline 0&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 0&1\\\hline\end{array}\\[6pt]
\begin{array}{|c|c|}\hline 0&0\\\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&0\\\hline 1&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 1&1\\\hline\end{array}\\[6pt]
\begin{array}{|c|c|}\hline 1&0\\\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&0\\\hline 1&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 1&1\\\hline\end{array}
\end{array}\right)
\end{array}
$$

It is easy to verify that

$$(1.2.10)\qquad \chi(x_{i_1 i_2}) = 4(i_1-1)+i_2,$$

i.e, we are counting local patterns in $\Sigma_{2\times 2}$ by going through each row successively in Table (1.2.9). Correspondingly, $\mathbf{X}_2$ can be referred to as an ordering matrix for $\Sigma_{2\times 2}$. Similarly, a 2×2 pattern can also be viewed as a vertical direct sum of two 2×1 patterns, i.e,

$$(1.2.11)\qquad y_{j_1 j_2} = y_{j_1}\oplus y_{j_2},$$

where

$$y_{j_l} = \left[\begin{array}{cc} u_{1l} & u_{2l}\end{array}\right] \quad or \quad \begin{array}{|c|c|}\hline u_{1l} & u_{2l}\\\hline\end{array}\ ,$$

and

$$(1.2.12)\qquad j_l = 1+2u_{1l}+u_{2l},$$

$1 \le l \le 2$. A 4×4 matrix $\mathbf{Y}_2 = [y_{j_1 j_2}]$ can also be obtained for $\Sigma_{2\times 2}$. i.e., we have

$$
\begin{array}{cc}
 & \begin{array}{cccc} \begin{array}{|c|c|}\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline\end{array} \end{array} \\
\begin{array}{c} \begin{array}{|c|c|}\hline 0&0\\\hline\end{array} \\[4ex] \begin{array}{|c|c|}\hline 0&1\\\hline\end{array} \\[4ex] \begin{array}{|c|c|}\hline 1&0\\\hline\end{array} \\[4ex] \begin{array}{|c|c|}\hline 1&1\\\hline\end{array} \end{array} &
\begin{pmatrix}
\begin{array}{|c|c|}\hline 0&0\\\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&0\\\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 0&0\\\hline\end{array} \\
\begin{array}{|c|c|}\hline 0&0\\\hline 0&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 0&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&0\\\hline 0&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 0&1\\\hline\end{array} \\
\begin{array}{|c|c|}\hline 0&0\\\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&0\\\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 1&0\\\hline\end{array} \\
\begin{array}{|c|c|}\hline 0&0\\\hline 1&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 1&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&0\\\hline 1&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 1&1\\\hline\end{array}
\end{pmatrix}
\end{array}
\tag{1.2.13}
$$

The relation between $\mathbf{X}_2$ and $\mathbf{Y}_2$ must be explored. Indeed, from (1.2.12), u_{kl} can be solved in terms of j_l, i.e., we have

$$u_{1l} = [\frac{j_l - 1}{2}] \tag{1.2.14}$$

and

$$u_{2l} = j_l - 1 - 2[\frac{j_l - 1}{2}], \tag{1.2.15}$$

where [ ] is the Gauss symbol, i.e., $[r]$ is the largest integer which is equal to or less than r. From (1.2.8), (1.2.12), (1.2.14) and (1.2.15), we have the following relations between indices i_1, i_2 and j_1, j_2.

$$j_1 = 1 + \sum_{k=1}^{2} [\frac{i_k - 1}{2}]\ 2^{2-k}, \tag{1.2.16}$$

$$j_2 = 1 + \sum_{k=1}^{2} \{\ i_k - 1 - 2\ [\frac{i_k - 1}{2}]\ \}\ 2^{2-k}, \tag{1.2.17}$$

and

$$i_1 = 1 + \sum_{l=1}^{2} [\frac{j_l - 1}{2}]\ 2^{2-l}, \tag{1.2.18}$$

$$i_2 = 1 + \sum_{l=1}^{2} \{\ j_l - 1 - 2\ [\frac{j_l - 1}{2}]\ \}\ 2^{2-l}. \tag{1.2.19}$$

From (1.2.16) and (1.2.17), (1.2.9) or $\mathbf{X}_2$ can also be represented by $y_{j_1 j_2}$ as

$$\mathbf{X}_2 = \begin{bmatrix} y_{11} & y_{12} & y_{21} & y_{22} \\ y_{13} & y_{14} & y_{23} & y_{24} \\ y_{31} & y_{32} & y_{41} & y_{42} \\ y_{33} & y_{34} & y_{43} & y_{44} \end{bmatrix}. \tag{1.2.20}$$

In (1.2.20), the indices $j_1 j_2$ are arranged by two Z-maps successively, as

$$\begin{bmatrix} 1 & \longrightarrow & 2 \\ & \swarrow & \\ 3 & \longrightarrow & 4 \end{bmatrix} \tag{1.2.21}$$

i.e., the path from 1 to 4 in (1.2.21) is Z shaped and is then called a Z-map. More precisely, $\mathbf{X}_2$ can be decomposed by

$$\mathbf{X}_2 = \begin{bmatrix} Y_{2;1} & Y_{2;2} \\ Y_{2;3} & Y_{2;4} \end{bmatrix} \tag{1.2.22}$$

and

$$Y_{2;k} = \begin{bmatrix} y_{k1} & y_{k2} \\ y_{k3} & y_{k4} \end{bmatrix}. \tag{1.2.23}$$

Where, $\mathbf{X}_2$ is arranged by a Z-map ($Y_{2;k}$) in (1.2.22) and each $Y_{2;k}$ is also arranged by a Z-map (y_{kl}) in (1.2.23). Therefore, the indices of y in (1.2.20) consist of two Z-maps.

The expression (1.2.20) of all local patterns in $\Sigma_{2\times 2}$ by y can be extended to all patterns in $\Sigma_{2\times n}$ for any $n \geq 3$. Indeed, a local pattern U in $\Sigma_{2\times n}$ can be viewed as the horizontal direct sum of two $1 \times n$ local patterns, i.e., U_1 and U_2, and also the vertical direct sums of n many 2×1 local patterns. As in (1.2.9), all patterns in $\Sigma_{2\times n}$ can be arranged by the ordering matrix

$$\mathbf{X}_n = [\ x_{n;i_1 i_2}\ ], \tag{1.2.24}$$

a $2^n \times 2^n$ matrix with entry $x_{n;i_1 i_2} = x_{n;i_1} \oplus x_{n;i_2}$, where $\chi(U_1) = i_1$ and $\chi(U_2) = i_2$ as in (1.2.4) and (1.2.5), $1 \leq i_1, i_2 \leq 2^n$. On the other hand, for two 2×2 patterns $y_{j_1 j_2}$ and $y_{j_2 j_3}$, we can attach them together to become a 2×3 pattern $y_{j_1 j_2 j_3}$, since the second row in $y_{j_1 j_2}$ and the first row of $y_{j_2 j_3}$ are identical, i.e.,

$$\begin{aligned} y_{j_1 j_2 j_3} &\equiv y_{j_1 j_2} \hat{\oplus} y_{j_2 j_3} \\ &\equiv y_{j_1} \oplus y_{j_2} \oplus y_{j_3}, \end{aligned} \tag{1.2.25}$$

Herein, a wedge direct sum $\hat{\oplus}$ is used for 2×2 patterns whenever they can be attached together. In this way, a $2\times n$ pattern $y_{j_1\cdots j_n}$ is obtained from $n-1$ many 2×2 patterns $y_{j_1j_2}, y_{j_2j_3}, \cdots, y_{j_{n-1}j_n}$ by

$$
\begin{aligned}
y_{j_1\cdots j_n} &\equiv y_{j_1j_2}\,\hat{\oplus}\, y_{j_2j_3}\,\hat{\oplus}\cdots\hat{\oplus}\, y_{j_{n-1}j_n} \\
&\equiv y_{j_1}\oplus y_{j_2}\oplus\cdots\oplus y_{j_n},
\end{aligned}
\tag{1.2.26}
$$

where $1\le j_k\le 4$, and $1\le k\le n$. Now, $\mathbf{X}_n$ in y expression can be obtained as follows.

Theorem 1.1. For any $n\ge 2$, $\Sigma_{2\times n}=\{y_{j_1\cdots j_n}\}$, where $y_{j_1\cdots j_n}$ is given in (1.2.26). Furthermore, the ordering matrix $\mathbf{X}_n$ can be decomposed by n Z-maps successively as

$$
\mathbf{X}_n=\begin{bmatrix} Y_{n;1} & Y_{n;2}\\ Y_{n;3} & Y_{n;4}\end{bmatrix},
\tag{1.2.27}
$$

$$
Y_{n;j_1\cdots j_k}=\begin{bmatrix} Y_{n;j_1\cdots j_k1} & Y_{n;j_1\cdots j_k2}\\ Y_{n;j_1\cdots j_k3} & Y_{n;j_1\cdots j_k4}\end{bmatrix},
\tag{1.2.28}
$$

for $1\le k\le n-2$, and

$$
Y_{n;j_1\cdots j_{n-1}}=\begin{bmatrix} y_{j_1\cdots j_{n-1}1} & y_{j_1\cdots j_{n-1}2}\\ y_{j_1\cdots j_{n-1}3} & y_{j_1\cdots j_{n-1}4}\end{bmatrix}.
\tag{1.2.29}
$$

Proof. From (1.2.12), (1.2.14) and (1.2.15), we have following table.

j_l	1	2	3	4
u_{1l}	0	0	1	1
u_{2l}	0	1	0	1

Table 2.1

For any $n\ge 2$, by (1.2.12),(1.2.14) and (1.2.15), it is easy to generalize (1.2.18) and (1.2.19) to

$$
i_{n;1}=1+\sum_{l=1}^{n}[\frac{j_l-1}{2}]2^{n-l},
\tag{1.2.30}
$$

and

$$i_{n;2} = 1 + \sum_{l=1}^{n} \{j_l - 1 - 2[\frac{j_l - 1}{2}]\}2^{n-l}. \tag{1.2.31}$$

From (1.2.30) and (1.2.31), we have

$$i_{n+1;1} = 2i_{n;1} - 1 + [\frac{j_{n+1} - 1}{2}], \tag{1.2.32}$$

and

$$i_{n+1;2} = 2i_{n;2} - 1 + \{j_{n+1} - 1 - 2[\frac{j_{n+1} - 1}{2}]\}. \tag{1.2.33}$$

Now, by induction on n the theorem follows from last two formulas and the table 2.1. The proof is complete. ■

Remark 1.2. The ordering matrix on $\Sigma_{m\times n}$ can also be introduced accordingly. However, when spatial entropy $h(\mathcal{B})$ of $\Sigma(\mathcal{B})$ is computed, only λ_n, the largest eigenvalue of $\mathbf{T}_n(\mathcal{B})$ must be known. Section 1.3 provides further details.

1.2.2 More Symbols on Larger Lattices

The idea introduced in the last section can be generalized to more symbols on $\mathbf{Z}_{m\times m}$, where $m \geq 3$. We first treat a case when m is even. Indeed, assume that $m = 2\ell$, $\ell \geq 2$ and $\mathcal{S}$ contains p elements. Now, we introduce the ordering matrices $\mathbf{X}_2 = [x_{i_1 i_2}]$ and $\mathbf{Y}_2 = [y_{j_1 j_2}]$ to $\Sigma_{2\ell\times 2\ell}$ as follows. Let $q = p^{\ell^2}$, $\mathbf{X}_2$ can be expressed by $y_{j_1 j_2}$, i.e.,

$$\mathbf{X}_2 = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_q \\ Y_{q+1} & Y_{q+2} & \cdots & Y_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{(q-1)q+1} & Y_{(q-1)q+2} & \cdots & Y_{q^2} \end{bmatrix}_{q\times q}, \tag{1.2.34}$$

with

$$Y_{j_1} = \begin{bmatrix} y_{j_1,1} & \cdots & y_{j_1,q} \\ y_{j_1,q+1} & \cdots & y_{j_1,2q} \\ \vdots & \ddots & \vdots \\ y_{j_1,(q-1)q+1} & \cdots & y_{j_1,q^2} \end{bmatrix}_{q\times q}. \tag{1.2.35}$$

Now, we can state recursion formulas for higher ordering matrix $\mathbf{X}_n = [x_{n;i_1 i_2}]_{q^n\times q^n}$ as follows and omit the proof for brevity.

Theorem 1.3. Suppose we have p symbols, $p \geq 2$ and let $q = p^{\ell^2}$, $\ell \geq 2$. For any $n \geq 2$, $\Sigma_{2\ell \times n\ell} = \{y_{j_1 j_2 \cdots j_n}\}$, where $y_{j_1 j_2 \cdots j_n} \equiv y_{j_1 j_2} \hat{\oplus} y_{j_2 j_3} \hat{\oplus} \cdots \hat{\oplus} y_{j_{n-1} j_n}$, $1 \leq j_k \leq q^2$ and $1 \leq k \leq n$. Furthermore, the ordering matrix $\mathbf{X}_n$ can be decomposed by n Z-maps successively as

$$\mathbf{X}_n = \begin{bmatrix} Y_{n;1} & Y_{n;2} & \cdots & Y_{n;q} \\ Y_{n;q+1} & Y_{n;q+2} & \cdots & Y_{n;2q} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{n;(q-1)q+1} & Y_{n;(q-1)q+2} & \cdots & Y_{n;q^2} \end{bmatrix} \tag{1.2.36}$$

$$Y_{n;j_1 \cdots j_k} =$$

$$\begin{bmatrix} Y_{n;j_1,\cdots,j_k,1} & Y_{n;j_1,\cdots,j_k,2} & \cdots & Y_{n;j_1,\cdots,j_k,q} \\ Y_{n;j_1,\cdots,j_k,q+1} & Y_{n;j_1,\cdots,j_k,q+2} & \cdots & Y_{n;j_1,\cdots,j_k,2q} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{n;j_1,\cdots,j_k,(q-1)q+1} & Y_{n;j_1,\cdots,j_k,(q-1)q+2} & \cdots & Y_{n;j_1,\cdots,j_k,q^2} \end{bmatrix} \tag{1.2.37}$$

for $1 \leq k \leq n-2$,

$$Y_{n;j_1 \cdots j_{n-1}} =$$

$$\begin{bmatrix} y_{j_1,\cdots,j_{n-1},1} & y_{j_1,\cdots,j_{n-1},2} & \cdots & y_{j_1,\cdots,j_{n-1},q} \\ y_{j_1,\cdots,j_{n-1},q+1} & y_{j_1,\cdots,j_{n-1},q+2} & \cdots & y_{j_1,\cdots,j_{n-1},2q} \\ \vdots & \vdots & \ddots & \vdots \\ y_{j_1,\cdots,j_{n-1},(q-1)q+1} & y_{j_1,\cdots,j_{n-1},(q-1)q+2} & \cdots & y_{j_1,\cdots,j_{n-1},q^2} \end{bmatrix}. \tag{1.2.38}$$

§ 1.3 Transition matrices

This section derives the transition matrices $\mathbf{T}_n$ for a given basic set $\mathcal{B}$. For simplicity, the study of two symbols $\mathcal{S} = \{0, 1\}$ on 2×2 lattice $\mathbf{Z}_{2\times 2}$ in two dimensional lattice space $\mathbf{Z}^2$ is of particular focus. The results can be extended to general cases.

1.3.1 2×2 systems

Given a basic set $\mathcal{B} \subset \Sigma_{2\times 2}$, horizontal and vertical transition matrices H_2 and V_2 can be defined by

$$H_2 = [h_{i_1 i_2}] \text{ and } V_2 = [v_{j_1 j_2}], \tag{1.3.1}$$

two 4×4 matrices with entries either 0 or 1 , according to following rules:

$$\begin{cases} h_{i_1 i_2} & = & 1 & if & x_{i_1 i_2} \in \mathcal{B}, \\ & = & 0 & if & x_{i_1 i_2} \in \Sigma_{2\times 2} - \mathcal{B}, \end{cases} \tag{1.3.2}$$

and

$$\begin{cases} v_{j_1 j_2} & = 1 & if & y_{j_1 j_2} \in \mathcal{B}, \\ & = 0 & if & y_{j_1 j_2} \in \Sigma_{2\times 2} - \mathcal{B}. \end{cases} \tag{1.3.3}$$

Obviously, $h_{i_1 i_2} = v_{j_1 j_2}$, where (i_1, i_2) and (j_1, j_2) are related according to (1.2.16)∼(1.2.19). Now, the transition matrix $\mathbf{T}_2$ for $\mathcal{B}$ can be defined by

$$\begin{aligned} \mathbf{T}_2 &\equiv \mathbf{T}_2(\mathcal{B}) \\ &= \begin{bmatrix} v_{11} & v_{12} & v_{21} & v_{22} \\ v_{13} & v_{14} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{41} & v_{42} \\ v_{33} & v_{34} & v_{43} & v_{44} \end{bmatrix}. \end{aligned} \tag{1.3.4}$$

Define

$$v_{j_1 j_2 \cdots j_n} = v_{j_1 j_2} \cdot v_{j_2 j_3} \cdots v_{j_{n-1} j_n}, \tag{1.3.5}$$

and

$$\mathbf{T}_n = [v_{j_1 j_2 \cdots j_n}],$$

then the transition matrix $\mathbf{T}_n$ for $\mathcal{B}$ defined on $\mathbf{Z}_{2\times n}$ is a $2^n \times 2^n$ matrix with entries $v_{j_1 \cdots j_n}$, which are either 1 or 0, by substituting $y_{j_1 \cdots j_n}$ by $v_{j_1 \cdots j_n}$ in $\mathbf{X}_n$, see (1.2.27)∼(1.2.29).

In the following, we give some interpretations for $\mathbf{T}_n$, one from an algebraic perspective and the other from Lindenmayer system (for details see Remark 1.5). For clarity, $\mathbf{T}_3$ can be written in a complete form as

$$\begin{bmatrix} v_{11}v_{11} & v_{11}v_{12} & v_{12}v_{21} & v_{12}v_{22} & v_{21}v_{11} & v_{21}v_{12} & v_{22}v_{21} & v_{22}v_{22} \\ v_{11}v_{13} & v_{11}v_{14} & v_{12}v_{23} & v_{12}v_{24} & v_{21}v_{13} & v_{21}v_{14} & v_{22}v_{23} & v_{22}v_{24} \\ v_{13}v_{31} & v_{13}v_{32} & v_{14}v_{41} & v_{14}v_{42} & v_{23}v_{31} & v_{23}v_{32} & v_{24}v_{41} & v_{24}v_{42} \\ v_{13}v_{33} & v_{13}v_{34} & v_{14}v_{43} & v_{14}v_{44} & v_{23}v_{33} & v_{23}v_{34} & v_{24}v_{43} & v_{24}v_{44} \\ v_{31}v_{11} & v_{31}v_{12} & v_{32}v_{21} & v_{32}v_{22} & v_{41}v_{11} & v_{41}v_{12} & v_{42}v_{21} & v_{42}v_{22} \\ v_{31}v_{13} & v_{31}v_{14} & v_{32}v_{23} & v_{32}v_{24} & v_{41}v_{13} & v_{41}v_{14} & v_{42}v_{23} & v_{42}v_{24} \\ v_{33}v_{31} & v_{33}v_{32} & v_{34}v_{41} & v_{34}v_{42} & v_{43}v_{31} & v_{43}v_{32} & v_{44}v_{41} & v_{44}v_{42} \\ v_{33}v_{33} & v_{33}v_{34} & v_{34}v_{43} & v_{34}v_{44} & v_{43}v_{33} & v_{43}v_{34} & v_{44}v_{43} & v_{44}v_{44} \end{bmatrix} \tag{1.3.6}$$

From an algebraic perspective, $\mathbf{T}_3$ can be defined through the classical Kronecker product (or tensor product) $\otimes$ and Hadamard product $\odot$. Indeed, for any two matrices $A = (a_{ij})$ and $B = (b_{kl})$, the Kronecker product of $A \otimes B$ is defined by

$$A \otimes B = (a_{ij}B). \tag{1.3.7}$$

On the other hand, for any two $n \times n$ matrices

$$C = (c_{ij}) \textit{ and } D = (d_{ij}),$$

where c_{ij} and d_{ij} are numbers or matrices. Then, Hadamard product of $C \odot D$ is defined by

$$C \odot D = (c_{ij} \cdot d_{ij}), \tag{1.3.8}$$

where the product $c_{ij} \cdot d_{ij}$ of c_{ij} and d_{ij} may be multiplication of numbers, numbers and matrices or matrices whenever it is well-defined. For instance, c_{ij} is number and d_{ij} is matrix.
Denoted by

$$\mathbf{T}_2 = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}, \tag{1.3.9}$$

where T_k is a 2×2 matrix with

$$T_k = \begin{bmatrix} v_{k1} & v_{k2} \\ v_{k3} & v_{k4} \end{bmatrix}. \tag{1.3.10}$$

Then, using Hadamard product, (1.3.6) can be written as

$$\mathbf{T}_3 = \begin{bmatrix} v_{11} & v_{12} & v_{21} & v_{22} \\ v_{13} & v_{14} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{41} & v_{42} \\ v_{33} & v_{34} & v_{43} & v_{44} \end{bmatrix} \odot \begin{bmatrix} T_1 & T_2 & T_1 & T_2 \\ T_3 & T_4 & T_3 & T_4 \\ T_1 & T_2 & T_1 & T_2 \\ T_3 & T_4 & T_3 & T_4 \end{bmatrix}, \tag{1.3.11}$$

and can also be written by Kronecker product with Hadamard product as

$$\mathbf{T}_3 = \left(\mathbf{T}_2 \right)_{4\times 4} \odot \left[\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \right], \tag{1.3.12}$$

where $(\mathbf{T}_2)_{4\times 4}$ is interpreted as a 4×4 matrix given as in (1.3.4). Hereinafter, $(M)_{k\times k}$ is used as the $k \times k$ matrix; its entries may also be matrices.

Furthermore, by (1.3.9) and (1.3.12), $\mathbf{T}_3$ can also be written as

$$\mathbf{T}_3 = \begin{bmatrix} T_1 \odot \mathbf{T}_2 & T_2 \odot \mathbf{T}_2 \\ T_3 \odot \mathbf{T}_2 & T_4 \odot \mathbf{T}_2 \end{bmatrix}. \tag{1.3.13}$$

Now, from the perspective of Lindenmayer system, (1.3.13) can be interpreted as a rewriting rule as follows:

To construct $\mathbf{T}_3$ from $\mathbf{T}_2$, simply replace T_k in (1.3.9) by $T_k \odot \mathbf{T}_2$, i.e,

$$T_k \longmapsto T_k \odot \mathbf{T}_2 = \begin{bmatrix} v_{k1}T_1 & v_{k2}T_2 \\ v_{k3}T_3 & v_{k4}T_4 \end{bmatrix}. \tag{1.3.14}$$

Now, $\mathbf{T}_3$ can be written as

$$\mathbf{T}_3 = \begin{bmatrix} v_{11}T_1 & v_{12}T_2 & v_{21}T_1 & v_{22}T_2 \\ v_{13}T_3 & v_{14}T_4 & v_{23}T_3 & v_{24}T_4 \\ v_{31}T_1 & v_{32}T_2 & v_{41}T_1 & v_{42}T_2 \\ v_{33}T_3 & v_{34}T_4 & v_{43}T_3 & v_{44}T_4 \end{bmatrix}. \tag{1.3.15}$$

Since v_{kj} is either 0 or 1. The entries of $\mathbf{T}_3$ in (1.3.15) are T_k, i.e, T_k can be taken as the "basic element" in constructing $\mathbf{T}_n$ $,n \geq 3$. As demonstrated later that(1.3.14) is an efficient means of constructing $\mathbf{T}_{n+1}$ from $\mathbf{T}_n$ for any $n \geq 2$.

Now, by induction on n, the following properties of transition matrix $\mathbf{T}_n$ on $\mathbf{Z}_{2\times n}$ can be easily proven.

Theorem 1.4. Let $\mathbf{T}_2$ be a transition matrix given by (1.3.4). Then, for higher order transition matrices $\mathbf{T}_n$, $n \geq 3$, we have the following three equivalent expressions
(I) $\mathbf{T}_n$ can be decomposed into n successive 2×2matrices (or n-successive Z-maps) as follows:

$$\mathbf{T}_n = \begin{bmatrix} T_{n;1} & T_{n;2} \\ T_{n;3} & T_{n;4} \end{bmatrix},$$

$$T_{n;j_1\cdots j_k} = \begin{bmatrix} T_{n;j_1\cdots j_k 1} & T_{n;j_1\cdots j_k 2} \\ T_{n;j_1\cdots j_k 3} & T_{n;j_1\cdots j_k 4} \end{bmatrix},$$

for $1 \leq k \leq n-2$ and

$$T_{n;j_1\cdots j_{n-1}} = \begin{bmatrix} v_{j_1\cdots j_{n-1}1} & v_{j_1\cdots j_{n-1}2} \\ v_{j_1\cdots j_{n-1}3} & v_{j_1\cdots j_{n-1}4} \end{bmatrix}.$$

Furthermore,

$$T_{n;k} = \begin{bmatrix} v_{k1}T_{n-1;1} & v_{k2}T_{n-1;2} \\ v_{k3}T_{n-1;3} & v_{k4}T_{n-1;4} \end{bmatrix}. \tag{1.3.16}$$

(II) Starting from

$$\mathbf{T}_2 = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix},$$

with

$$T_k = \begin{pmatrix} v_{k1} & v_{k2} \\ v_{k3} & v_{k4} \end{pmatrix},$$

$\mathbf{T}_n$ can be obtained from $\mathbf{T}_{n-1}$ by replacing T_k by $T_k \odot \mathbf{T}_2$ according to (1.3.14).

(III)

$$\mathbf{T}_n = (\mathbf{T}_{n-1})_{2^{n-1}\times 2^{n-1}} \odot \left(E_{2^{n-2}} \otimes \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \right),$$

where E_{2^k} is the $2^k \times 2^k$ matrix with 1 as its entries.

Proof.

(I)The proof is simply replaced $Y_{n;j_1\cdots j_k}$ and $y_{j_1\cdots j_n}$ by $T_{n;j_1\cdots j_k}$ and $v_{j_1\cdots j_n}$ in Theorem 2.1, respectively.

(II) follow from (I) directly.

(III) follow from (I), we have

$$\mathbf{T}_n = \begin{bmatrix} T_{n;1} & T_{n;2} \\ T_{n;3} & T_{n;4} \end{bmatrix}.$$

And by (1.3.16), we get following formula.

$$\begin{aligned} \mathbf{T}_n &= \begin{bmatrix} v_{11}T_{n;1} & v_{12}T_{n;2} & v_{21}T_{n;1} & v_{22}T_{n;2} \\ v_{13}T_{n;3} & v_{14}T_{n;4} & v_{23}T_{n;3} & v_{24}T_{n;4} \\ v_{31}T_{n;1} & v_{32}T_{n;2} & v_{41}T_{n;1} & v_{42}T_{n;2} \\ v_{33}T_{n;3} & v_{34}T_{n;4} & v_{43}T_{n;3} & v_{44}T_{n;4} \end{bmatrix} \\ &= (\mathbf{T}_{n-1})_{2^{n-1}\times 2^{n-1}} \odot \left(E_{2^{n-2}} \otimes \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \right). \end{aligned}$$

The proof is complete. ■

Remark 1.5. While studying the growth processes of plants, Lindenmayer, e.g.[39], derived a developmental algorithm, i.e., a set of rules which describes plant development in time. Thereafter, a system with a set of rewriting rules was called Lindenmayer system or L-system. From Theorem 1.4(III), the family of transition matrices $\{\mathbf{T}_n\}_{n\geq 2}$ is a two-dimensional L-system with a rewriting rule(1.3.16). Similar to many L-systems, our system $\mathbf{T}_n$ also enjoys the simplicity of recursion formulas and self-similarity.

As for spatial entropy $h(\mathcal{B})$, we have the following theorem.

Theorem 1.6. Given a basic set $\mathcal{B} \subset \Sigma_{2\times 2}$, let λ_n be the largest eigenvalue of the associated transition matrix $\mathbf{T}_n$ which is defined in Theorem 1.4. Then,

$$h(\mathcal{B}) = \lim_{n\to\infty} \frac{\log \lambda_n}{n}. \tag{1.3.17}$$

Proof. By the same arguments as in [13], the limit (1.1.2) is well-defined and exists. From the construction of $\mathbf{T}_n$, we observe that for $m \geq 2$,

$$\begin{aligned} \Gamma_{m\times n}(\mathcal{B}) &= \sum_{1\leq i,j\leq 2^n} (\mathbf{T}_n^{m-1})_{i,j} \\ &\equiv \#(\mathbf{T}_n^{m-1}). \end{aligned} \tag{1.3.18}$$

As in a one dimensional case, we have

$$\lim_{m\to\infty} \frac{\log \#(\mathbf{T}_n^{m-1})}{m} = \log \lambda_n,$$

e.g. [42]. Therefore,

$$\begin{aligned} h(\mathcal{B}) &= \lim_{m,n\to\infty} \frac{\log \Gamma_{m\times n}(\mathcal{B})}{mn} \\ &= \lim_{n\to\infty} \frac{1}{n}\left(\lim_{m\to\infty} \frac{\log \Gamma_{m\times n}(\mathcal{B})}{m}\right) \\ &= \lim_{n\to\infty} \frac{\log \lambda_n}{n}. \end{aligned}$$

The proof is complete. ∎

1.3.2 Computation of Maximum Eigenvalues and Spatial Entropy

Given a transition matrix $\mathbf{T}_2$, for any $n \geq 2$, the characteristic polynomials $|\mathbf{T}_n - \lambda|$ are of degree 2^n. In general, computing or estimating the largest eigenvalue $\lambda_n = \lambda_n(\mathbf{T}_2)$ of $|\mathbf{T}_n - \lambda|$ for a large n is relatively difficult. However, in this section, we present a class of $\mathbf{T}_2$ in which $\lambda_n(\mathbf{T}_2)$ can be computed explicitly. Indeed, assume that $\mathbf{T}_2$ has the form of $\begin{bmatrix} A & B \\ B & A \end{bmatrix}$ in (1.3.9), i.e.,

$$T_1 = T_4 = A = \begin{bmatrix} a & a_2 \\ a_3 & a \end{bmatrix}, \tag{1.3.19}$$

and

$$T_2 = T_3 = B = \begin{bmatrix} b & b_2 \\ b_3 & b \end{bmatrix}, \tag{1.3.20}$$

where a, a_2, a_3, b, b_2 and b_3 are either 0 or 1.

We need the following lemma.

Lemma 1.7. Let A and B be non-negative and non-zero $m \times m$ matrices, respectively, and α and β are positive numbers. The maximum eigenvalue of $\begin{bmatrix} A & \alpha B \\ \beta B & A \end{bmatrix}$ is then the maximum eigenvalue of

$$A + \sqrt{\alpha\beta}B.$$

Proof. Consider

$$\begin{vmatrix} A-\lambda & \alpha B \\ \beta B & A-\lambda \end{vmatrix} = 0.$$

For $|A-\lambda| \neq 0$, the last equation is equivalent to

$$\begin{vmatrix} A-\lambda & B \\ 0 & (A-\lambda) - \alpha\beta B(A-\lambda)^{-1}B \end{vmatrix} = 0,$$

or

$$|I - \alpha\beta((A-\lambda)^{-1}B)^2| = 0.$$

Then, we have

$$|A + \sqrt{\alpha\beta}B - \lambda| = 0 \;\; or \;\; |A - \sqrt{\alpha\beta}B - \lambda| = 0.$$

Since A and B are non-negative and α and β are positive, verifying that the maximum eigenvalue λ of $\begin{bmatrix} A & \alpha B \\ \beta B & A \end{bmatrix}$ and $A + \sqrt{\alpha\beta}B$ are equal is relatively easy. The proof is complete. ■

Now, we can state our computation results for $\lambda_n(\mathbf{T}_2)$ when $\mathbf{T}_2$ satisfies (1.3.19) and (1.3.20).

Theorem 1.8. Assume that $\mathbf{T}_2 = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$ and $A = \begin{bmatrix} a & a_2 \\ a_3 & a \end{bmatrix}$ and $B = \begin{bmatrix} b & b_2 \\ b_3 & b \end{bmatrix}$ where $a, b, a_2, a_3, b_2, b_3 \in \{0, 1\}$. For $n \geq 2$, let λ_n be the largest eigenvalue of

$$|\mathbf{T}_n - \lambda| = 0.$$

Then

$$\lambda_n = \alpha_{n-1} + \beta_{n-1}, \tag{1.3.21}$$

where α_k and β_k satisfy the following recursion relations:

$$\alpha_{k+1} = a\alpha_k + b\beta_k, \tag{1.3.22}$$

$$\beta_{k+1} = \sqrt{(a_2\alpha_k + b_2\beta_k)(a_3\alpha_k + b_3\beta_k)}, \tag{1.3.23}$$

for $k \geq 0$, and

$$\alpha_0 = \beta_0 = 1. \tag{1.3.24}$$

Furthermore, the spatial entropy $h(\mathbf{T}_2)$ is equal to $\log \xi_*$, where ξ_* is the maximum root of the following polynomials $Q(\xi)$:
(I) if $a_2 = a_3 = 1$,

$$\begin{aligned} Q(\xi) \equiv \quad & 4\xi^2(\xi - a)^2 + (\gamma^2 - 4\delta)(\xi - a)^2 \\ & -\gamma^2\xi^2 - 2\gamma(2b - a\gamma)\xi - (2b - a\gamma)^2, \end{aligned} \tag{1.3.25}$$

where

$$\gamma = b_2 + b_3 \textit{ and } \delta = b_2 b_3. \tag{1.3.26}$$

(II) if $a_2a_3 = 0$ and $a_2b_3 + a_3b_2 = 1$,

$$Q(\xi) \equiv \xi^3 - a\xi^2 - \delta\xi + a\delta - b. \tag{1.3.27}$$

Moreover, if $a_2a_3 = 0$ and $a_2b_3 + a_3b_2 = 0$, then $h(\mathbf{T}_2) = 0$.

Proof. Owing to the special structure of $\mathbf{T}_2$, it is easy to verify that for any $k \geq 2$, we have

$$\mathbf{T}_k = \begin{bmatrix} A_k & B_k \\ B_k & A_k \end{bmatrix},$$

and

$$\mathbf{T}_{k+1} = \begin{bmatrix} A_{k+1} & B_{k+1} \\ B_{k+1} & A_{k+1} \end{bmatrix},$$

here

$$A_{k+1} = \mathbf{T}_k \odot A = \begin{bmatrix} aA_k & a_2B_k \\ a_3B_k & aA_k \end{bmatrix}, \tag{1.3.28}$$

and

$$B_{k+1} = \mathbf{T}_k \odot B = \begin{bmatrix} bA_k & b_2B_k \\ b_3B_k & bA_k \end{bmatrix}, \tag{1.3.29}$$

$A_2 = A$ and $B_2 = B$. Now by Lemma 1.7,

$$|\mathbf{T}_{n+1} - \lambda_{n+1}| = 0,$$

implies

$$|A_{n+1} + B_{n+1} - \lambda_{n+1}| = 0. \tag{1.3.30}$$

Let

$$\alpha_0 = 1 \;\; and \;\; \beta_0 = 1.$$

By induction on k, $1 \leq k \leq n$, and using (1.3.28),(1.3.29),(1.3.30) and Lemma 1.7, it is straight forward to derive

$$|\alpha_k A_{n-k+1} + \beta_k B_{n-k+1} - \lambda_{n+1}| = 0, \tag{1.3.31}$$

with α_k and β_k satisfy (1.3.22) and (1.3.23). In particular,

$$\alpha_n = a\alpha_{n-1} + b\beta_{n-1}, \tag{1.3.32}$$

$$\beta_n = \{(a_2\alpha_{n-1} + b_2\beta_{n-1})(a_3\alpha_{n-1} + b_3\beta_{n-1})\}^{\frac{1}{2}}, \tag{1.3.33}$$

and

$$\lambda_{n+1} = \alpha_n + \beta_n.$$

This proves the first part of the theorem.
The remainder of the proof, demonstrates that $h(\mathbf{T}_2) = \log \lambda_*$ where λ_* is the maximum root of $Q(\lambda)$. From (1.3.33), we have

$$\begin{aligned} \beta_n^2 = a_2a_3\alpha_{n-1}^2 \;&+\; (a_2b_3 + a_3b_2)\alpha_{n-1}\beta_{n-1} \\ &+\; b_2b_3\beta_{n-1}^2. \end{aligned} \tag{1.3.34}$$

Now, in (1.3.34), we first solve α_{n-1} in terms of β_{n-1} and β_n, then substitute α_{n-1} and α_n into (1.3.32) to obtain difference equations involving β_{n+1}, β_n and β_{n-1}. There are two cases:

Case I. If $a_2 = a_3 = 1$, then we have

$$\alpha_{n-1} = \frac{1}{2}\{-\gamma\beta_{n-1} + (4\beta_n^2 + (\gamma^2 - 4\delta)\beta_{n-1}^2)^{\frac{1}{2}}\}. \tag{1.3.35}$$

Substituting (1.3.35) into (1.3.32), yields

$$
\begin{aligned}
(4\beta_{n+1}^2 + (\gamma^2 - 4\delta)\beta_n^2)^{\frac{1}{2}} = \gamma\beta_n + (2b - a\gamma)\beta_{n-1} \\
+ a(4\beta_n^2 + (\gamma^2 - 4\delta)\beta_{n-1}^2)^{\frac{1}{2}}.
\end{aligned} \tag{1.3.36}
$$

Now, let

$$\xi_n = \frac{\beta_n}{\beta_{n-1}}, \tag{1.3.37}$$

and after dividing (1.3.36) by β_{n-1}, we have

$$\xi_n\{4\xi_{n+1}^2 + (\gamma^2 - 4\delta)\}^{\frac{1}{2}} = \gamma\xi_n + (2b - a\gamma) + a\{4\xi_n^2 + (\gamma^2 - 4\delta)\}^{\frac{1}{2}}. \tag{1.3.38}$$

(1.3.38) can be written as the following iteration map:

$$\xi_{n+1} = G_1(\xi_n), \tag{1.3.39}$$

where

$$G_1(\xi) = \frac{1}{2}\{4\delta + 2\gamma g(\xi) + g^2(\xi)\}^{\frac{1}{2}}, \tag{1.3.40}$$

and

$$g(\xi) = (2b - a\gamma)\xi^{-1} + a\{4 + (\gamma^2 - 4\delta)\xi^{-2}\}^{\frac{1}{2}}. \tag{1.3.41}$$

We first observe the fixed point ξ_* of $G_1(\xi)$, i.e., $\xi_* = G(\xi_*)$, is a root of $Q(\xi)$. Indeed, by letting $\xi_n = \xi_{n+1} = \xi_*$ in (1.3.38), we have

$$(\xi_* - a)(4\xi_*^2 + (\gamma^2 - 4\delta))^{\frac{1}{2}} = \gamma\xi_* + (2b - a\gamma),$$

which gives us $Q(\xi_*) = 0$.

It can be proven that the maximum fixed point of $G_1(\xi)$ or the maximum root ξ_* of $Q(\xi) = 0$ satisfies $1 \le \xi_* \le 2$ and

$$\xi_n \to \xi_* \quad as \quad n \to \infty. \tag{1.3.42}$$

Details are omitted here for brevity. By (1.3.21), (1.3.35) and (1.3.37), we can also prove that

$$\frac{\lambda_{n+1}}{\lambda_n} \to \xi_* \quad as \quad n \to \infty. \tag{1.3.43}$$

Hence, $h(\mathbf{T}_2) = \log \xi_*$.

Case II. If $a_2a_3 = 0$ and $a_2b_3 + a_3b_2 = 1$,
then, from (1.3.33), we have

$$\alpha_{n-1} = \beta_n^2\beta_{n-1}^{-1} - \delta\beta_{n-1}. \tag{1.3.44}$$

Again, substituting (1.3.44) into (1.3.32) and letting (1.3.37) lead to

$$\xi_{n+1}^2\xi_n - a\xi_n^2 - \delta\xi_n + a\delta - b = 0, \tag{1.3.45}$$

i.e.,

$$\xi_{n+1} = G_2(\xi_n),$$

where

$$G_2(\xi) = \{a\xi + \delta + (b - a\delta)\xi^{-1}\}^{\frac{1}{2}}. \tag{1.3.46}$$

The maximum fixed point ξ_* of (1.3.46) is the maximum root of $Q(\xi) = 0$ in (1.3.27). It can also be proven that (1.3.42) and (1.3.43) holds in this case.

Finally, if $a_2a_3 = 0$ and $a_2b_3 + a_3b_2 = 0$, then β_n are all equal for $n \geq 1$. Hence, α_n is at most linear growth in n, implying that $h(\mathbf{T}_2) = 0$. The proof is thus complete. ■

For completeness, we list all $\mathbf{T}_2$ which satisfy (1.3.19) and (1.3.20) and have positive entropy $h(\mathbf{T}_2)$. The table is arranged based on the magnitude of $h(T_2)$. The polynomial $Q(.)$ in either (1.3.25) or (1.3.27) has been simplified whenever possible.

	A	B	$Q(\lambda)$	λ_*
(1)	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\lambda - 2$	2
(2)	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *or* $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$	$\lambda^3 - 2\lambda^2 + \lambda - 1$	(i)
(3)(α)	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *or* $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\lambda^2 - \lambda - 1$	g
(3)(β)	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\lambda^2 - \lambda - 1$	g
(3)(γ)	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\lambda^2 - \lambda - 1$	g
(4)	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$	$\lambda^3 - \lambda^2 - 1$	(ii)
	$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$		
(5)	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *or* $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\lambda^3 - \lambda - 1$	(iii)
(6)	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *or* $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$	$\lambda^4 - \lambda - 1$	(iv)

(i) $\lambda_* \doteq 1.75488$, (ii) $\lambda_* \doteq 1.46557$, (iii) $\lambda_* \doteq 1.32472$, (iv) $\lambda_* \doteq 1.22074$ where, $g \doteq 1.61803$, is the golden mean, a root of $\lambda^2 - \lambda - 1 = 0$.

Table 1.1

The recursion formulas for λ_n are

(1)		$\lambda_n = 2^n,$
(2)		$\lambda_{n+1} = \lambda_n + (\lambda_n\lambda_{n-1})^{\frac{1}{2}},$
(3)	(α)	$\lambda_{n+1} = \lambda_n + (\lambda_n(\lambda_n - \lambda_{n-1}))^{\frac{1}{2}},$
	(β)	$\lambda_{n+1} = \lambda_n + \lambda_{n-1},$
	(γ)	$\lambda_{n+1} = \lambda_n + \lambda_{n-1},$
(4)		$\lambda_{n+1} = \lambda_n + (\lambda_{n-1}(\lambda_n - \lambda_{n-1}))^{\frac{1}{2}},$
(5)		$\lambda_{n+1} = (\lambda_n\beta_{n-1})^{\frac{1}{2}} + \beta_{n-1},$ where $\beta_{n-1} = \lambda_n - \lambda_{n-1} + \cdots + (-1)^n,$
(6)		$\lambda_{n+1} = \lambda_n + (\lambda_n\beta_{n-2})^{\frac{1}{2}} - \beta_{n-2}.$

Table 1.2

Remark 1.9.
(i) According to Table 1.2, for cases (1)$\sim$(4), λ_{n+1} depends only on two preceding terms, λ_n and λ_{n-1}. However, in (5) and (6), λ_{n+1} depends on all of its preceding terms $\lambda_1, \cdots, \lambda_n$.
(ii) From Lemma 1.7 and Theorem 1.8, in addition to the maximum eigenvalue we can obtain a complete set of eigenvalues of $\mathbf{T}_n$ explicitly.
(iii) In Theorem 1.8, polynomial $Q(\xi)$ given in (1.3.25) or (1.3.27) is the limiting equation for $\lambda_n^{\frac{1}{n}}$. It is interesting to know is there any limiting equation for general $\mathbf{T}_n$.

Remark 1.10. Similar to the concept in Theorem 1.8, if $\mathbf{T}_2$ does not satisfy (1.3.19) and (1.3.20), another special structure can allow us to obtain explicit recursion formulas of λ_n and compute its spatial entropy $h(\mathbf{T}_2)$ explicitly.

1.3.3 $2\ell \times 2\ell$ Systems

The results in last two subsections can be generalized to p-symbols on $\mathbf{Z}_{2\ell\times 2\ell}$. Given a basic set $\mathcal{B} \subset \Sigma_{2\ell\times 2\ell}$, horizontal and vertical transition matrices $H_2 = [h_{i_1i_2}]_{q^2\times q^2}$ and $V_2 = [v_{j_1j_2}]_{q^2\times q^2}$, where $q = p^{\ell^2}$, can be defined according the rules (1.3.2) and (1.3.3) by replacing $\Sigma_{2\times 2}$ with $\Sigma_{2\ell\times 2\ell}$, respectively. Then the transition matrix $\mathbf{T}_2(\mathcal{B})$ for $\mathcal{B}$ can be defined by

$$\mathbf{T}_2 = \mathbf{T}_2(\mathcal{B}) = \begin{bmatrix} V_1 & V_2 & \cdots & V_q \\ V_{q+1} & V_{q+2} & \cdots & V_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ V_{(q-1)q+1} & V_{(q-1)q+2} & \cdots & V_{q^2} \end{bmatrix} \tag{1.3.47}$$

where

$$V_m = \begin{bmatrix} v_{m,1} & v_{m,2} & \cdots & v_{m,q} \\ v_{m,(q+1)} & v_{m,q+2} & \cdots & v_{m,2q} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m,(q-1)q+1} & v_{m,(q-1)q+2} & \cdots & v_{m,q^2} \end{bmatrix}, \tag{1.3.48}$$

$1 \le m \le q^2$. The higher order transition matrix $\mathbf{T}_n = [v_{j_1 j_2 \cdots j_n}]$ for $\mathcal{B}$ defined on $\mathbf{Z}_{2\ell\times n\ell}$ is a $q^n \times q^n$ matrix, where $v_{j_1 j_2 \cdots j_n}$ is given by (1.3.5) which are either 1 or 0, by substituting $y_{j_1\cdots j_n}$by $v_{j_1\cdots j_n}$ in $\mathbf{X}_n$, see (1.2.36)$\sim$(1.2.38). For completeness, we state the following theorem for $\mathbf{T}_n$ and omit the proof for brevity.

Theorem 1.11. Let $\mathbf{T}_2$ be a transition matrix given by (1.3.47) and (1.3.48). Then for higher order transition matrices $\mathbf{T}_n$, $n \ge 3$, we have the following three equivalent expressions

(I) $\mathbf{T}_n$ can be decomposed into n successive $q \times q$ matrices as follows:

$$\mathbf{T}_n = \begin{bmatrix} T_{n;1} & \cdots & T_{n;q} \\ T_{n;q+1} & \cdots & T_{n;2q} \\ \vdots & \ddots & \vdots \\ T_{n;(q-1)q+1} & \cdots & T_{n;q^2} \end{bmatrix}$$

$$T_{n;j_1\cdots j_k} = \begin{bmatrix} T_{n;j_1,\cdots,j_k,1} & \cdots & T_{n;j_1,\cdots,j_k,q} \\ T_{n;j_1,\cdots,j_k,q+1} & \cdots & T_{n;j_1,\cdots,j_k,2q} \\ \vdots & \ddots & \vdots \\ T_{n;j_1,\cdots,j_k,(q-1)q+1} & \cdots & T_{n;j_1,\cdots,j_k,q^2} \end{bmatrix}$$

for $1 \le k \le n-2$ and

$$T_{n;j_1\cdots j_{n-1}} = \begin{bmatrix} v_{j_1,\cdots,j_{n-1},1} & \cdots & v_{j_1,\cdots,j_{n-1},q} \\ v_{j_1,\cdots,j_{n-1},q+1} & \cdots & v_{j_1,\cdots,j_{n-1},2q} \\ \vdots & \ddots & \vdots \\ v_{j_1,\cdots,j_{n-1},(q-1)q+1} & \cdots & v_{j_1,\cdots,j_{n-1},q^2} \end{bmatrix}.$$

Furthermore,

$$T_{n;k} = \begin{bmatrix} v_{k,1}T_{n-1;1} & \cdots & v_{k,q}T_{n-1;q} \\ v_{k,q+1}T_{n-1;q+1} & \cdots & v_{k,2q}T_{n-1;2q} \\ \vdots & \ddots & \vdots \\ v_{k,(q-1)q+1}T_{n-1;(q-1)q+1} & \cdots & v_{k,q^2}T_{n-1;q^2} \end{bmatrix}$$

(II) Starting from

$$\mathbf{T}_2 = \begin{bmatrix} T_1 & \cdots & T_q \\ T_{q+1} & \cdots & T_{2q} \\ \vdots & \ddots & \vdots \\ T_{(q-1)q+1} & \cdots & T_{q^2} \end{bmatrix},$$

with

$$T_k = \begin{bmatrix} v_{k,1} & \cdots & v_{k,q} \\ v_{k,q+1} & \cdots & v_{k,2q} \\ \vdots & \ddots & \vdots \\ v_{k,(q-1)q+1} & \cdots & v_{k,q^2} \end{bmatrix},$$

$\mathbf{T}_n$ can be obtained from $\mathbf{T}_{n-1}$ by replacing T_k by $T_k \odot \mathbf{T}_2$ according to

$$T_k \mapsto T_k \odot \mathbf{T}_2 = \begin{bmatrix} v_{k,1}T_1 & \cdots & v_{k,q}T_q \\ v_{k,q+1}T_{q+1} & \cdots & v_{k,2q}T_{2q} \\ \vdots & \ddots & \vdots \\ v_{k,(q-1)q+1}T_{(q-1)q+1} & \cdots & v_{k,q^2}T_{q^2} \end{bmatrix}$$

(III)

$$\mathbf{T}_n = (\mathbf{T}_{n-1})_{q^{n-1}\times q^{n-1}} \odot (E_{q^{n-2}} \otimes \mathbf{T}_2).$$

For the spatial entropy $h(\mathcal{B})$, we have a similar result as in Theorem 1.6.

Theorem 1.12. Given a basic set $\mathcal{B} \subset \Sigma_{m_1\times m_2}$, let ℓ be the smallest integer such that $2\ell \geq m_1$ and $2\ell \geq m_2$, and let $\widetilde{\mathcal{B}} = \Sigma_{2\ell\times 2\ell}(\mathcal{B})$. Suppose $\lambda_{n;\ell}$ be the largest eigenvalue of the associated transition matrix $\mathbf{T}_n$, which is defined in Theorem 1.11. Then

$$h(\mathcal{B}) = \frac{1}{\ell^2}\lim_{n\to\infty}\frac{log\lambda_{n;\ell}}{n}.$$

Proof. As in Theorem 1.6,

$$\begin{aligned}
h(\mathcal{B}) &= \lim_{m,n\to\infty} \frac{log\Gamma_{m\ell\times n\ell}(\widetilde{\mathcal{B}})}{m\ell\times n\ell} \\
&= \frac{1}{\ell^2}\lim_{n\to\infty}\frac{1}{n}(\lim_{m\to\infty}\frac{log\#(T_n^{m-1}(\widetilde{\mathcal{B}}))}{m}) \\
&= \frac{1}{\ell^2}\lim_{n\to\infty}\frac{1}{n}(\lim_{m\to\infty}\frac{log\lambda_{n;\ell}^{m-1}}{m}) \\
&= \frac{1}{\ell^2}\lim_{n\to\infty}\frac{log\lambda_{n;\ell}}{n}.
\end{aligned}$$

The proof is complete. ■

1.3.4 Relation with Matrix Shifts

Under many circumstances, we are given a pair of horizontal transition matrix $H=(h_{ij})_{p\times p}$ and vertical transition matrix $V=(v_{ij})_{p\times p}$, where h_{ij} and $v_{ij}\in\{0,1\}$, e.g. [13, 29, 30, 32]. Now, the set of all admissible patterns which can be generated by H and V on $\mathbf{Z}_{m_1\times m_2}$ and $\mathbf{Z}^2$ are denoted by $\Sigma_{m_1\times m_2}(H;V)$ and $\Sigma(H;V)$, respectively. Furthermore, $\Sigma_{m_1\times m_2}(H;V)$ and $\Sigma(H;V)$ can be characterized by

(1.3.49)
$$\begin{aligned}
\Sigma_{m_1\times m_2}(H;V) = \{U\in\Sigma_{m_1\times m_2,p} : h_{u_\alpha u_{\alpha+e_1}}=1 \textit{ and } v_{u_\beta u_{\beta+e_2}}=1, \\
\textit{where } e_1=(1,0),\ e_2=(0,1),\ \alpha=(\alpha_1,\alpha_2),\ \beta=(\beta_1,\beta_2) \\
\textit{with } 1\le\alpha_1\le m_1-1\ ,\ 1\le\alpha_2\le m_2 \textit{ and } 1\le\beta_1\le m_1\ 1\le\beta_2\le m_2-1\}
\end{aligned}$$

and

$$\begin{aligned}
\Sigma(H;V)=\{U\in\Sigma_p^2 : h_{u_\alpha u_{\alpha+e_1}}=1 \textit{ and } v_{u_\beta u_{\beta+e_2}}=1 \\
\textit{for all } \alpha,\beta\in\mathbf{Z}^2\}.
\end{aligned} \tag{1.3.50}$$

In literature, $\Sigma(H;V)$ is often called Matrix shift, Markov shift or subshift of finite types, e.g. [13, 30, 32, 38]

As before, we are concerned about constructing $\Sigma_{m_1\times m_2}(H;V)$. We first show that the established theories can be applied to answer this question. Indeed, we introduce $\mathcal{S}=\{0,1,2,\cdots,p-1\}$. On $\mathbf{Z}_{2\times2}$, consider local pattern $U=(u_{\alpha_1\alpha_2})$ with $u_{\alpha_1\alpha_2}\in\ \mathcal{S}$. Define the ordering matrices $\mathbf{X}_2=[x_{i_1i_2}]_{p^2\times p^2}$ and $\mathbf{Y}_2=[y_{j_1j_2}]_{p^2\times p^2}$ for $\Sigma_{2\times2}$. Now, the basic set $\mathcal{B}(H;V)$ determined by H and V can be expressed as

(1.3.51)
$$\mathcal{B}(H;V)=\{U=(u_{\alpha_1\alpha_2})\in\Sigma_{2\times2} : h_{u_{11}u_{21}}h_{u_{12}u_{22}}v_{u_{11}u_{12}}v_{u_{21}u_{22}}=1\}.$$

Therefore, the transition matrix $\mathbf{T}_2 = \mathbf{T}_2(H;V)$ can be expressed as $\mathbf{T}_2 = [t_{j_1 j_2}]_{p^2 \times p^2}$ with $t_{j_1 j_2} = 1$ if and only if $y_{j_1 j_2} \in \mathcal{B}(H;V)$, i.e., $t_{j_1 j_2} = 1$ if and only if

$$h_{u_{11}u_{21}} h_{u_{12}u_{22}} v_{u_{11}u_{12}} v_{u_{21}u_{22}} = 1, \tag{1.3.52}$$

where j_l is related to $u_{\alpha_1 \alpha_2}$ according to (1.2.12) similarly.

Now, $\mathbf{T}_n = \mathbf{T}_n(H;V)$ can be constructed recursively from $\mathbf{T}_2(H;V)$ by Theorem 1.11. Then λ_n and spatial entropy $h(H;V)$ can be studied by Theorem 1.12. It is easy to verify $\mathbf{T}_n(H;V) = \overline{\mathbf{T}}^{(n)}_{H,V}$, the transition matrix obtained by Juang et al in [30]. Furthermore, $T^{(n)}_{H,V}$ in [30] can also be obtained by deleting the rows and columns formed by zeros in $\mathbf{T}_n(H;V)$.

On the other hand, given a basic set $\mathcal{B} \subset \Sigma_{2\times 2,p}$ (or $\Sigma_{2l\times 2l,p}$), in general there is no horizontal transition matrix $H = (h_{ij})_{p\times p}$ and vertical transition matrix $V = (v_{ij})_{p\times p}$ such that $\mathcal{B} = \mathcal{B}(H;V)$ given by (1.3.51). Indeed, the number of subsets of $\Sigma_{2\times 2,p}$ is 2^{p^4} and the number of $\mathcal{B}(H;V)$ is at most 2^{2p^2} and $2^{2p^2} < 2^{p^4}$ for any $p \geq 2$. However, as mentioned in p.468[38], one can recode any shift of finite type to a matrix subshift.

Notably, the n-th order transition matrix $\mathbf{T}_n(\mathcal{B})$ is a $q^n \times q^n$ matrix with $q = p^{\ell^2}$ and the n-th order transition matrix $\mathbf{T}_n(H(\mathcal{B}); V(\mathcal{B})))$ generated by $\mathbf{T}_2(H(\mathcal{B}); V(\mathcal{B})))$ is a $m^n \times m^n$ matrix. Consequently, if $m = \#\mathcal{B}$ is relatively small compared with $q = p^{l^2}$, we may study the eigenvalue problems of $\mathbf{T}_n(H(\mathcal{B}); V(\mathcal{B}))$. It is clear, small m generates less admissible patterns and then smaller entropy. For $\mathcal{B}$ with positive entropy $h(\mathcal{B})$ as in Table 3.1, $\#\mathcal{B}$ is much larger than $q = 2$. Therefore, in general working on $\mathbf{T}_n(\mathcal{B})$ is better than on $\mathbf{T}_n(H(\mathcal{B}); V(\mathcal{B})))$.

Reference

[1] J.C. Ban, K.P. Chien, S.S. Lin, Spatial disorder of CNN-with asymmetric output funtion, International J. of Bifurcation and Chaos, 11(2001), pp. 2085-2095.

[2] J.C. Ban, C.H. Hsu, S.S. Lin, Spatial disorder of Cellular Neural Network-with biased term, International J. of Bifurcation and Chaos, 12(2002), pp. 525-534.

[3] J.C. Ban, S.S. Lin, Y.H. Lin, Patterns Generation and Transition Matrices in Higher Dimensional Lattice Models, preprint (2004).

[4] P.W. Bates, A. Chmaj, A discrete convolution model for phase transitions, Arch. Rat. Mech. Anal, 150(1999), pp. 281-305

[5] P.W. Bates, K. Lu, B. Wang, Attractors for lattice dynamical systems, International J. of Bifurcation and Chaos, 11(2001),pp. 143-153.

[6] J.C. Ban, S.S. Lin, C.W. Shih, Exact number of mosaic patterns in cellular neural networks International J. of Bifurcation and Chaos, 11(2001), pp. 1645-1653.

[7] J. Bell, Some threshold results for modes of myelinated nerves, Math. Biosci., 54 (1981), pp. 181-190.

[8] J. Bell, C. Cosner, Threshold behavior and propagation for nonlinear differential-difference systems motivated by modeling myelinated axons, Quart. Appl. Math., 42(1984), pp. 1-14.

[9] J.W. Cahn, Theory of crystal growth and interface motion in crystalline materials, Acta Metallurgica, 8(1960), pp. 554-562.

[10] S.S Chen, C.W Shih, Asymptotic behaviors in a transiently chaotic neural network, Discrete Contin. Dynam. Systems, 10(2004), pp. 805-836.

[11] S.N. Chow, J. Mallet-paret, Pattern formation and spatial chaos in lattice dynamical systems II, IEEE Trans. Circuits Systems, 42(1995), pp. 752-756.

[12] S.N. Chow, J. Mallet-paret, E.S. Van vleck, Dynamics of lattice differential equations, International J. of Bifurcation and Chaos, 9(1996), pp. 1605-1621.

[13] S.N. Chow, J. Mallet-paret, E.S. Van vleck, Pattern formation and spatial chaos in spatially discrete evolution equations, Random Comput. Dynam., 4(1996), pp. 109-178.

[14] S.N. Chow, W. Shen, Dynamics in a discrete Nagumo equation: Spatial topological chaos, SIAM J. Appl. Math, 55(1995), pp. 1764-1781.

[15] L.O. Chua, CNN: A paradigm for complexity. World Scientific Series on Nonlinear Science, Series A,31. World Scietific, Singapore.(1998)

[16] L.O. Chua, K. R. Crounse, M. Hasler, P. Thiran, Pattern formation properties of autonomous cellular neural networks, IEEE Trans. Circuits Systems, 42(1995), pp. 757-774.

[17] L.O. Chua, T. Roska, The CNN paradigm, IEEE Trans. Circuits Systems, 40(1993), pp. 147-156.

[18] L.O. Chua, L. Yang, Cellular neural networks: Theory, IEEE Trans. Circuits Systems, 35(1988), pp. 1257-1272.

[19] L.O. Chua, L. Yang, Cellular neural networks: Applications, IEEE Trans. Circuits Systems, 35(1988), pp. 1273-1290.

[20] H.E. Cook, D.De Fontaine, J.E. Hilliard, A model for diffusion on cubic lattices and its application to the early stages of ordering, Acta Metallurgica, 17(1969), pp. 765-773.

[21] G.B. Ermentrout, Stable periodic solutions to discrete and continuum arrays of weakly coupled nonlinear oscillators, SIAM J. Appl. Math., 52 (1992), pp. 1665-1687.

[22] G.B. Ermentrout, N. Kopell, Inhibition-produced patterning in chains of coupled nonlinear oscillators, SIAM J. Appl. Math., 54 (1994), pp. 478-507.

[23] G.B. Ermentrout, N. Kopell, T.L. Williams, On chains of oscillators forced at one end, SIAM J. Appl. Math., 51(1991), pp.1397-1417.

[24] T. Eveneux, J.P. Laplante, Propagation failure in arrays of coupled bistable chemical reactors, J. Phys. Chem., 96(1992), pp. 4931-4934.

[25] W.J. Firth, Optical memory and spatial chaos, Phys. Rev. Lett., 61(1988), pp. 329-332.

[26] M. Hillert, A solid-solution model for inhomogeneous systems, Acta Metallurgica, 9(1961), pp.525-535.

[27] C.H. Hsu, J. Juang, S.S. Lin, W.W. Lin, Cellular neural networks: local patterns for general template, International J. of Bifurcation and Chaos, 10(2000), pp.1645-1659.

[28] C.H. Hsu, S.S. Lin, Traveling waves in lattice dynamical systems, J. Differential Equations, 164(2000),pp.431-450.

[29] J. Juang, S.S. Lin, Cellular Neural Networks: Mosaic pattern and spatial chaos, SIAM J. Appl. Math., 60(2000), pp.891-915, 2000.

[30] J. Juang, S.S. Lin, W.W. Lin, S.F. Shieh, Two dimensional spatial entropy, International J. of Bifurcation and Chaos, 10(2000), pp.2845-2852.

[31] J. Juang, S.S. Lin, Cellular Neural Networks: Defect pattern and spatial chaos, preprint.

[32] J. Juang, S.F. Shieh, On the spatial entropy of two-dimensional golden mean, International J. of Bifurcation and Chaos, 14(2004), pp.309-319.

[33] J.P. Keener, Propagation and its failure in coupled systems of discrete excitable cells, SIAM J. Appl. Math., 47(1987), pp.556-572.

[34] J.P. Keener, The effects of discrete gap junction coupling on propagation in myocardium, J. Theor. Biol., 148(1991), pp.49-82.

[35] A.L. Kimball, A. Varghese, R.L. Winslow, Simulating cardiac sinus and atrial network dynamics on the connection machine, Phys. D, 64(1993), pp.281-298.

[36] S.S. Lin, T.S. Yang, Spatial entropy of one dimensional celluar neural network, International J. of Bifurcation and Chaos, 10(2000), pp.2129-2140.

[37] S.S. Lin, T.S. Yang, On the spatial entropy and patterns of two-dimensional cellular neural network, International J. of Bifurcation and Chaos, 12(2002), pp.115-128.

[38] D. Lind, B. Marcus, An introduction to symbolic dynamics and coding, Cambrige University Press, New York, 1995.

[39] A. Lindenmayer, P. Prusinkiewicz, The algorithmic beauty of plants, Springer-Verlag, New York, 1990.

[40] R.S. Mackay, J.A. Sepulchre, Multistability in networks of weakly coupled bistable units, Phys. D, 82(1995), pp. 243-254.

[41] J. Mallet-paret, S.N. Chow, Pattern formation and spatial chaos in lattice dynamical systems I, IEEE Trans. Circuits Systems, 42(1995), pp. 746-751.

[42] C. Robinson, Dynamical Systems, CRC Press, London (1995).

[43] W. Shen, Lifted lattices, hyerbolic structures, and topological disorders in coupled map lattices, SIAM J. Appl. Math, 56(1996), 1379-1399.

Chapter 2

Patterns Generation and Spatial Entropy in Two-dimensional Lattice Models

§ 2.1 Introduction

Lattices are important in scientifically modelling underlying spatial structures. Investigations in this field have covered phase transition [11], [12], [34], [35], [36], [37], [38], [45], [46], [47], [48], chemical reaction [7], [8], [24], biology [9], [10], [21], [22], [23], [31], [32], [33] and image processing and pattern recognition [16], [17], [18], [19], [20], [25]. In the field of lattice dynamical systems (LDS) and cellular neural networks (CNN), the complexity of the set of all global patterns recently attracted substantial interest. In particular, its spatial entropy has received considerable attention [1],[2], [3], [4], [5], [13], [14], [15], [28], [29],[30], [39], [40], [41], [42], [43], [44].

The one dimensional spatial entropy h can be found from an associated transition matrix $\mathbb{T}$. The spatial entropy h equals $\log \rho(\mathbb{T})$, where $\rho(\mathbb{T})$ is the maximum eigenvalue of $\mathbb{T}$.

In two-dimensional situations, higher transition matrices have been discovered in [30] and developed systematically [4] by studying the patterns generation problem.

This study extends our previous work [4]. For simplicity, two symbols on 2×2 lattice $\mathbb{Z}_{2\times 2}$ are considered. A transition matrix in the horizontal (or vertical) direction

$$\mathbb{A}_2 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \tag{2.1.1}$$

which is linked to a set of admissible local patterns on $\mathbb{Z}_{2\times 2}$ is considered, where $a_{ij} \in \{0,1\}$ for $1 \le i, j \le 4$. The associated vertical (or horizontal) transition matrix $\mathbb{B}_2$ is given by

$$\mathbb{B}_2 = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} \tag{2.1.2}$$

$\mathbb{A}_2$ and $\mathbb{B}_2$ are connected to each other as follows.

$$\mathbb{A}_2 = \left[\begin{array}{cc|cc} b_{11} & b_{12} & b_{21} & b_{22} \\ b_{13} & b_{14} & b_{23} & b_{24} \\ \hline b_{31} & b_{32} & b_{41} & b_{42} \\ b_{33} & b_{34} & b_{43} & b_{44} \end{array}\right] = \begin{bmatrix} A_{2;1} & A_{2;2} \\ A_{2;3} & A_{2;4} \end{bmatrix}, \tag{2.1.3}$$

and

$$\mathbb{B}_2 = \left[\begin{array}{cc|cc} a_{11} & a_{12} & a_{21} & a_{22} \\ a_{13} & a_{14} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{41} & a_{42} \\ a_{33} & a_{34} & a_{43} & a_{44} \end{array}\right] = \begin{bmatrix} B_{2;1} & B_{2;2} \\ B_{2;3} & B_{2;4} \end{bmatrix}. \tag{2.1.4}$$

Notably if $\mathbb{A}_2$ represents the horizontal (or vertical) transition matrix then $\mathbb{B}_2$ represents the vertical (or horizontal) transition matrix. Results that hold for $\mathbb{A}_2$ are also valid for $\mathbb{B}_2$. Therefore, for simplicity, only $\mathbb{A}_2$ is presented herein.

The recursive formulae for n-th order transition matrices $\mathbb{A}_n$ defined on $\mathbb{Z}_{2\times n}$ were obtained [4] as follows

$$\mathbb{A}_{n+1} = \begin{bmatrix} b_{11}A_{n;1} & b_{12}A_{n;2} & b_{21}A_{n;1} & b_{22}A_{n;2} \\ b_{13}A_{n;3} & b_{14}A_{n;4} & b_{23}A_{n;3} & b_{24}A_{n;4} \\ b_{31}A_{n;1} & b_{32}A_{n;2} & b_{41}A_{n;1} & b_{42}A_{n;2} \\ b_{33}A_{n;3} & b_{34}A_{n;4} & b_{43}A_{n;3} & b_{44}A_{n;4} \end{bmatrix} \tag{2.1.5}$$

whenever

$$\mathbb{A}_n = \begin{bmatrix} A_{n;1} & A_{n;2} \\ A_{n;3} & A_{n;4} \end{bmatrix}, \tag{2.1.6}$$

for $n \geq 2$, or equivalently,

$$A_{n+1;\alpha} = \begin{bmatrix} b_{\alpha 1}A_{n;1} & b_{\alpha 2}A_{n;2} \\ b_{\alpha 3}A_{n;3} & b_{\alpha 4}A_{n;4} \end{bmatrix}, \tag{2.1.7}$$

for $\alpha \in \{1, 2, 3, 4\}$. The number of all admissible patterns defined on $\mathbb{Z}_{m\times n}$ which can be generated from $\mathbb{A}_2$ is now defined by

(2.1.8)
$$\begin{aligned}\Gamma_{m,n}(\mathbb{A}_2) &= |\mathbb{A}_n^{m-1}| \\ &= \text{the summation of all entries in } 2^n \times 2^n \text{ matrix } \mathbb{A}_n^{m-1}.\end{aligned}$$

The spatial entropy $h(\mathbb{A}_2)$ is defined as

$$h(\mathbb{A}_2) = \lim_{m,n\to\infty} \frac{1}{mn} \log \Gamma_{m,n}(\mathbb{A}_2) = \lim_{m,n\to\infty} \frac{1}{mn} \log |\mathbb{A}_n^{m-1}|. \tag{2.1.9}$$

The existence of the limit (2.1.9) has been shown in [4], [15], [30]. When $h(\mathbb{A}_2) > 0$, the number of admissible patterns grows exponentially with the lattice size $m \times n$. In this situation, spatial chaos arises. When $h(\mathbb{A}_2) = 0$, pattern formation occurs.

To compute the double limit in (2.1.9), $n \geq 2$ can be fixed initially and m allowed to tend to infinite [30] and [4]; then the Perron-Frobenius theorem is applied;

$$\lim_{m\to\infty} \frac{1}{m} \log |\mathbb{A}_n^{m-1}| = \log \rho(\mathbb{A}_n), \tag{2.1.10}$$

which implies

$$h(\mathbb{A}_2) = \lim_{n\to\infty} \frac{1}{n} \log \rho(\mathbb{A}_n), \tag{2.1.11}$$

where $\rho(M)$ is the maximum eigenvalue of matrix M. $\mathbb{A}_n$ is a $2^n \times 2^n$ matrix, so computing $\rho(\mathbb{A}_n)$ is usually quite difficult when n is larger. Moreover, (2.1.11) does not produce any error estimation in the estimated sequence $\frac{1}{n} \log \rho(\mathbb{A}_n)$ and its limit $h(\mathbb{A}_2)$. This causes a serious problem in computing the entropy. However, for a class of $\mathbb{A}_2$, the recursive formulae for $\rho(\mathbb{A}_n)$ can be discovered, along with a limiting equation to $\rho^* = \exp(h(\mathbb{A}_2))$, as in [4].

This study takes a different approach to resolve these difficulties. Previously, the double limit (2.1.9) was initially examined by taking the m-limit firstly as in (2.1.10). Now, for each fixed $m \geq 2$, the n-limit in (2.1.9) is studied. Therefore, the limit

$$\lim_{n\to\infty} \frac{1}{n} \log |\mathbb{A}_n^{m-1}| \tag{2.1.12}$$

is considered. Write

$$\mathbb{A}_n^m = \begin{bmatrix} A_{m,n;1} & A_{m,n;2} \\ A_{m,n;3} & A_{m,n;4} \end{bmatrix}. \tag{2.1.13}$$

The investigation of (2.1.12) would be simpler if a recursive formula such as (2.1.7) could be found for $A_{m,n;\alpha}$. The first task in this study is to solve this problem. For matrix multiplication, the indices of $A_{n;\alpha},\ \alpha \in \{1,2,3,4\}$ are conveniently expressed as

$$\mathbb{A}_n = \begin{bmatrix} A_{n;11} & A_{n;12} \\ A_{n;21} & A_{n;22} \end{bmatrix}. \tag{2.1.14}$$

Then

$$A_{m,n;\alpha} = \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n;\alpha}, \tag{2.1.15}$$

where

$$A^{(k)}_{m,n;\alpha} = A_{n;j_1j_2}A_{n;j_2j_3}\cdots A_{n;j_mj_{m+1}}, \tag{2.1.16}$$

$$k = 1 + \sum_{i=2}^{m} 2^{m-i}(j_i - 1), \tag{2.1.17}$$

and

$$\alpha = 2(j_1 - 1) + j_{m+1}. \tag{2.1.18}$$

$A^{(k)}_{m,n;\alpha}$ in (2.1.16) is called an elementary pattern of order (m,n), and is a fundamental element in constructing $A_{m,n;\alpha}$ in (2.1.15). Notably the elementary patterns are in lexicographic order, according to (2.1.17). As in [4], the following m-th order ordering matrix.

$$\mathbb{X}_{m,n} = \begin{bmatrix} X_{m,n;1} & X_{m,n;2} \\ X_{m,n;3} & X_{m,n;4} \end{bmatrix}, \tag{2.1.19}$$

is represented to record systematically these elementary patterns, where

$$X_{m,n;\alpha} = (A^{(k)}_{m,n;\alpha})^t_{1\le k\le 2^{m-1}} \tag{2.1.20}$$

is a 2^{m-1} column vector.

The first main result of this study is to introduce the connecting operator $\mathbb{C}_m$, and to use it to derive a recursive formula like (2.1.7) for $A^{(k)}_{m,n;\alpha}$. Indeed,

$$\mathbb{C}_m = \begin{bmatrix} C_{m;11} & C_{m;12} & C_{m;13} & C_{m;14} \\ C_{m;21} & C_{m;22} & C_{m;23} & C_{m;24} \\ C_{m;31} & C_{m;32} & C_{m;33} & C_{m;34} \\ C_{m;41} & C_{m;42} & C_{m;43} & C_{m;44} \end{bmatrix} \tag{2.1.21}$$

$$= \begin{bmatrix} S_{m;11} & S_{m;12} & S_{m;21} & S_{m;22} \\ S_{m;13} & S_{m;14} & S_{m;23} & S_{m;24} \\ S_{m;31} & S_{m;32} & S_{m;41} & S_{m;42} \\ S_{m;33} & S_{m;34} & S_{m;43} & S_{m;44} \end{bmatrix}, \tag{2.1.22}$$

where
(2.1.23)

$$\begin{aligned} C_{m;ij} &= \left(\left[\begin{array}{cc} a_{i1} & a_{i2} \\ a_{i3} & a_{i4} \end{array} \right] \circ \left(\hat{\otimes} \left[\begin{array}{cc} B_{2;1} & B_{2;2} \\ B_{2;3} & B_{2;4} \end{array} \right]^{m-2} \right)_{2\times 2} \right)_{2^{m-1}\times 2^{m-1}} \\ &\circ \left(E_{2^{m-2}\times 2^{m-2}} \otimes \left[\begin{array}{cc} a_{1j} & a_{2j} \\ a_{3j} & a_{4j} \end{array} \right] \right)_{2^{m-1}\times 2^{m-1}} \end{aligned}$$

is a $2^{m-1} \times 2^{m-1}$ matrix where $E_{k\times k}$ is the $k \times k$ full matrix; $\otimes$ denotes the Kronecker product, $\circ$ denotes the Hadamard product and the product $\hat{\otimes}$ which involves both the Kronecker product and the Hadamard product, as stipulated by Definition 2.2.

In Theorem 2.4, $C_{m;ij}$ is shown to be $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_m i_{m+1}}$, with $i_1 = i$ and $i_{m+1} = j$. Therefore, all admissible paths of $\mathbb{A}_2$ from i to j with length m are arranged systematically in matrix $C_{m;ij}$. Now, the recursive formula is

$$A^{(k)}_{m,n+1;\alpha} = \begin{bmatrix} \sum_{l=1}^{2^{m-1}} (S_{m;\alpha 1})_{kl} A^{(l)}_{m,n;1} & \sum_{l=1}^{2^{m-1}} (S_{m;\alpha 2})_{kl} A^{(l)}_{m,n;2} \\ \sum_{l=1}^{2^{m-1}} (S_{m;\alpha 3})_{kl} A^{(l)}_{m,n;3} & \sum_{l=1}^{2^{m-1}} (S_{m;\alpha 4})_{kl} A^{(l)}_{m,n;4} \end{bmatrix}, \tag{2.1.24}$$

for $m \geq 2,\ n \geq 2,\ 1 \leq k \leq 2^{m-1}$ and $1 \leq \alpha \leq 4$. (2.1.24) is the generalization of (2.1.7).

The recursive formula (2.1.24) immediately yields a lower bound on entropy. Indeed, for any positive integer K and diagonal periodic cycle $\beta_1\beta_2 \cdots \beta_K \beta_{K+1}$, where $\beta_j \in \{1, 4\}$ and $\beta_{K+1} = \beta_1$,

$$h(\mathbb{A}_2) \geq \frac{1}{mK} \log \rho(S_{m;\beta_1\beta_2} S_{m;\beta_2\beta_3} \cdots S_{m;\beta_K \beta_{K+1}}). \tag{2.1.25}$$

Equation (2.1.25) implies $h(\mathbb{A}_2) > 0$, if a diagonal periodic cycle of $\beta_1\beta_2 \cdots \beta_K \beta_1$ applies, with a maximum eigenvalue of $S_{m;\beta_1\beta_2} \cdots S_{m;\beta_K\beta_1}$ that greater than one. This method powerfully yields the positivity of spatial entropy, which is hard in examining the complexity of patterns generation problems.

However, the subadditivity of $\Gamma_{m,n}(\mathbb{A}_2)$ is known to imply

$$h(\mathbb{A}_2) \leq \frac{1}{mn} \log \Gamma_{m,n}(\mathbb{A}_2) \tag{2.1.26}$$

as in [15]. Consequently, (2.1.8), (2.1.10) and (2.1.26) indicate an upper bound of entropy as

$$h(\mathbb{A}_2) \leq \frac{1}{n} \log \rho(\mathbb{A}_n), \tag{2.1.27}$$

for any $n \geq 2$.

However, the Perron-Frobenius theorem also implies

$$\limsup_{m\to\infty} \frac{1}{m} \log tr(\mathbb{A}_n^{m-1}) = \log \rho(\mathbb{A}_n), \tag{2.1.28}$$

where $tr(M)$ denotes the trace of matrix M [26], [27]. Therefore, (2.1.28) implies

$$h(\mathbb{A}_2) = \limsup_{m,n\to\infty} \frac{1}{mn} \log tr(\mathbb{A}_n^{m-1}). \tag{2.1.29}$$

In studying the double-limit of (2.1.29), for each fixed $m \geq 2$, the n-limit in (2.1.29)

$$\limsup_{n\to\infty} \frac{1}{n} \log tr(\mathbb{A}_n^{m-1}) \tag{2.1.30}$$

is first considered. (2.1.30) can be studied by introducing the following trace operator

$$\mathbb{T}_m = \begin{bmatrix} C_{m;11} & C_{m;22} \\ C_{m;33} & C_{m;44} \end{bmatrix}. \tag{2.1.31}$$

Then, a recursive formula for $tr(\mathbb{A}_n^m)$ can be verified

$$tr(\mathbb{A}_n^m) = \left| \mathbb{T}_m^{n-2} \begin{pmatrix} trX_{m,2;1} \\ \\ trX_{m,2;4} \end{pmatrix} \right|, \tag{2.1.32}$$

for $n \geq 2$, where $tr(X_{m,n;\alpha}) = (trA_{m,n;\alpha}^{(k)})^t_{1\leq k\leq 2^{m-1}}$ and $|v| = \sum_{j=1}^{l} v_j$ for vector $v = (v_1, \cdots, v_l)^t$. Consequently, (2.1.29) and (2.1.32) yield

$$h(\mathbb{A}_2) = \limsup_{m\to\infty} \frac{1}{m} \log \rho(\mathbb{T}_m). \tag{2.1.33}$$

Notably, for a large class of $\mathbb{A}_2$, the limit sup in (2.1.28), (2.1.29), (2.1.30) and (2.1.33) can be replaced by limit. See section 3.3 for details.

Now, (2.1.33) can be applied to find the upper bounds of entropy. For example, when $\mathbb{A}_2$ is symmetric,

$$h(\mathbb{A}_2) \leq \frac{1}{2m} \log \rho(\mathbb{T}_{2m}), \tag{2.1.34}$$

for any $m \geq 1$. Since

$$\mathbb{T}_n \leq \mathbb{B}_n \tag{2.1.35}$$

can be shown for any $n \geq 2$. Generally, (2.1.33) and (2.1.34) yield better approximation than (2.1.11) and (2.1.27).

In summary, this study yields lower-bound estimates of entropy like (2.1.25) by introducing connecting operators $\mathbb{C}_m$, and upper-bound estimates of entropy like (2.1.34) by introducing trace operators $\mathbb{T}_m$. This approach accurately and effectively yields the spatial entropy.

The rest of this paper is organized as follows. Section 3.2 derives the connecting operator $\mathbb{C}_m$ which can recursively reduce higher order elementary patterns to patterns of lower order. Then, the lower-bound of spatial entropy can be found by computing the maximum eigenvalues of the diagonal periodic cycles of sequence $S_{m;\alpha\beta}$. Section 3.3 addresses the trace operator $\mathbb{T}_m$ of $\mathbb{C}_m$. The entropy can be calculated by computing the maximum eigenvalues of $\mathbb{T}_m$. When $\mathbb{A}_2$ is symmetric, the upper-bounds of entropy are also found. Section 3.4 briefly discusses the theory for many symbols on larger lattices.

§ 2.2 Connecting Operators

2.2.1 Connecting operators and ordering matrices

This section derives connecting operators and investigates their properties. For clarity, two symbols on 2×2 lattice $\mathbb{Z}_{2\times 2}$ are examined first. Section 3.4 addresses more general situations.

Let $\mathbb{A}_2$ and $\mathbb{B}_2$ be defined as in (3.1.1)∼(2.1.4). The column matrices $\widetilde{\mathbb{A}_2}$ and $\widetilde{\mathbb{B}_2}$ of $\mathbb{A}_2$ and $\mathbb{B}_2$ are defined by

$$\widetilde{\mathbb{A}_2} = \left[\begin{array}{cc|cc} a_{11} & a_{21} & a_{12} & a_{22} \\ a_{31} & a_{41} & a_{32} & a_{42} \\ \hline a_{13} & a_{23} & a_{14} & a_{24} \\ a_{33} & a_{43} & a_{34} & a_{44} \end{array}\right] = \begin{bmatrix} \tilde{A}_{2;1} & \tilde{A}_{2;2} \\ \tilde{A}_{2;3} & \tilde{A}_{2;4} \end{bmatrix} \tag{2.2.1}$$

and

$$\widetilde{\mathbb{B}_2} = \left[\begin{array}{cc|cc} b_{11} & b_{21} & b_{12} & b_{22} \\ b_{31} & b_{41} & b_{32} & b_{42} \\ \hline b_{13} & b_{23} & b_{14} & b_{24} \\ b_{33} & b_{43} & b_{34} & b_{44} \end{array}\right] = \left[\begin{array}{cc} \tilde{B}_{2;1} & \tilde{B}_{2;2} \\ \tilde{B}_{2;3} & \tilde{B}_{2;4} \end{array}\right] \tag{2.2.2}$$

, respectively.

For matrices of higher order $n \geq 2$, $\mathbb{A}_n$, $\mathbb{A}_{n+1}$ and $A_{n+1;\alpha}$ are defined as in (2.1.5)∼(2.1.7).

For matrix multiplication, the indices of $A_{n;\alpha}$ are conveniently expressed as

$$\mathbb{A}_n = \left[\begin{array}{cc} A_{n;11} & A_{n;12} \\ A_{n;21} & A_{n;22} \end{array}\right]. \tag{2.2.3}$$

Clearly, $A_{n;\alpha} = A_{n;j_1j_2}$, where

$$\alpha = \alpha(j_1, j_2) = 2(j_1 - 1) + j_2. \tag{2.2.4}$$

For $m \geq 2$, the elementary pattern in the entries of $\mathbb{A}_n^m$ is represented by

$$A_{n;j_1j_2}A_{n;j_2j_3}\cdots A_{n;j_mj_{m+1}},$$

where $j_s \in \{1, 2\}$. A lexicographic order for multiple indices

$$J_{m+1} = (j_1j_2\cdots j_mj_{m+1})$$

is introduced, using

$$\chi(J_{m+1}) = 1 + \sum_{s=2}^{m} 2^{m-s}(j_s - 1). \tag{2.2.5}$$

Now,

$$A^{(k)}_{m,n;\alpha} = A_{n;j_1j_2}A_{n;j_2j_3}\cdots A_{n;j_mj_{m+1}}, \tag{2.2.6}$$

where

$$\alpha = \alpha(j_1, j_{m+1}) = 2(j_1 - 1) + j_{m+1} \tag{2.2.7}$$

and

$$k = \chi(J_{m+1}) \tag{2.2.8}$$

is given in (2.2.5). Notably, (2.2.5) and (2.2.8) do not involve j_{m+1} but (2.2.7)does.

Therefore, $\mathbb{A}_n^m$ can be expressed as

$$\mathbb{A}_n^m = \begin{bmatrix} A_{m,n;1} & A_{m,n;2} \\ A_{m,n;3} & A_{m,n;4} \end{bmatrix}, \tag{2.2.9}$$

where

$$A_{m,n;\alpha} = \sum_{k=1}^{2^{m-1}} A_{m,n;\alpha}^{(k)}. \tag{2.2.10}$$

Furthermore,

$$X_{m,n;\alpha} = (A_{m,n;\alpha}^{(k)})^t_{1\le k\le 2^{m-1}}. \tag{2.2.11}$$

$1 \le k \le 2^{m-1}$, $X_{m,n;\alpha}$ is a 2^{m-1} column-vector that consists of all elementary patterns in $A_{m,n;\alpha}$. The ordering matrix $\mathbb{X}_{m,n}$ of $\mathbb{A}_n^m$ is now defined by

$$\mathbb{X}_{m,n} = \begin{bmatrix} X_{m,n;1} & X_{m,n;2} \\ X_{m,n;3} & X_{m,n;4} \end{bmatrix}. \tag{2.2.12}$$

The ordering matrix $\mathbb{X}_{m,n}$ allows the elementary patterns to be tracked during the reduction from $\mathbb{A}_{n+1}^m$ to $\mathbb{A}_n^m$. This careful book-keeping provides a systematic way to generate the admissible patterns and later, lower-bound estimates of spatial entropy.

The following simplest example is studied first to illustrate the above concept.

Example 2.1. For $m = 2$, the following can easily be verified;

$$\mathbb{A}_n^2 = \begin{bmatrix} A_{n;11}^2 + A_{n;12}A_{n;21} & A_{n;11}A_{n;12} + A_{n;12}A_{n;22} \\ A_{n;21}A_{n;11} + A_{n;22}A_{n;21} & A_{n;21}A_{n;12} + A_{n;22}^2 \end{bmatrix}, \tag{2.2.13}$$

and

$$\left.\begin{array}{ll} A_{2,n;1}^{(1)} = A_{n;11}^2, & A_{2,n;1}^{(2)} = A_{n;12}A_{n;21}, \\ A_{2,n;2}^{(1)} = A_{n;11}A_{n;12}, & A_{2,n;2}^{(2)} = A_{n;12}A_{n;22}, \\ A_{2,n;3}^{(1)} = A_{n;21}A_{n;11}, & A_{2,n;3}^{(2)} = A_{n;22}A_{n;21}, \\ A_{2,n;4}^{(1)} = A_{n;21}A_{n;12}, & A_{2,n;4}^{(2)} = A_{n;22}^2. \end{array}\right\}. \tag{2.2.14}$$

Therefore,

$$\left.\begin{array}{l} X_{2,n;1} = \begin{bmatrix} A_{n;11}^2 \\ A_{n;12}A_{n;21} \end{bmatrix}, \quad X_{2,n;2} = \begin{bmatrix} A_{n;11}A_{n,12} \\ A_{n;12}A_{n;22} \end{bmatrix}, \\ \\ X_{2,n;3} = \begin{bmatrix} A_{n;21}A_{n;11} \\ A_{n;22}A_{n;21} \end{bmatrix}, \quad X_{2,n;4} = \begin{bmatrix} A_{n;21}A_{n,12} \\ A_{n;22}^2 \end{bmatrix}. \end{array}\right\}. \tag{2.2.15}$$

Applying (2.1.7), and by a straightforward computation,

$$X_{2,n+1;1} = \begin{bmatrix} A^2_{n+1;11} \\ A_{n+1;12}A_{n+1;21} \end{bmatrix} \tag{2.2.16}$$

$$= \begin{bmatrix} \begin{bmatrix} b^2_{11}A^2_{n;1} + b_{12}b_{13}A_{n;2}A_{n;3} & b_{11}b_{12}A_{n;1}A_{n;2} + b_{12}b_{14}A_{n;2}A_{n;4} \\ b_{13}b_{11}A_{n;3}A_{n;1} + b_{14}b_{13}A_{n;4}A_{n;3} & b_{13}b_{12}A_{n;3}A_{n;2} + b^2_{14}A^2_{n;4} \end{bmatrix} \\ \\ \begin{bmatrix} b_{21}b_{31}A^2_{n;1} + b_{22}b_{33}A_{n;2}A_{n;3} & b_{21}b_{32}A_{n;1}A_{n;2} + b_{22}b_{34}A_{n;2}A_{n;4} \\ b_{23}b_{31}A_{n;3}A_{n;1} + b_{24}b_{33}A_{n;4}A_{n;3} & b_{23}b_{32}A_{n;3}A_{n;2} + b_{24}b_{34}A^2_{n;4} \end{bmatrix} \end{bmatrix}$$

Clearly, the j_1j_2 entries of $A^2_{n+1;11}$ and $A_{n+1;12}A_{n+1;21}$ in (3.2.9) consist of entries of $X_{2,n;\alpha}$ in (3.2.7) with $\alpha = \alpha(j_1, j_2)$ in (2.2.4). Moreover, the terms in (3.2.9) can be rearranged in terms of $X_{2,n;\alpha}$ by exchanging the second row in the first matrix with the first row in the second matrix in (3.2.9) as follows.
(2.2.17)

$$\begin{bmatrix} \begin{bmatrix} b^2_{11} & b_{12}b_{13} \\ b_{21}b_{31} & b_{22}b_{33} \end{bmatrix} \begin{bmatrix} A^2_{n;1} \\ A_{n;2}A_{n;3} \end{bmatrix} & \begin{bmatrix} b_{11}b_{12} & b_{12}b_{14} \\ b_{21}b_{32} & b_{22}b_{34} \end{bmatrix} \begin{bmatrix} A_{n;1}A_{n;2} \\ A_{n;2}A_{n;4} \end{bmatrix} \\ \\ \begin{bmatrix} b_{13}b_{11} & b_{14}b_{13} \\ b_{23}b_{31} & b_{24}b_{33} \end{bmatrix} \begin{bmatrix} A_{n;3}A_{n;1} \\ A_{n;4}A_{n;3} \end{bmatrix} & \begin{bmatrix} b_{13}b_{12} & b^2_{14} \\ b_{23}b_{32} & b_{24}b_{34} \end{bmatrix} \begin{bmatrix} A_{n;3}A_{n;2} \\ A^2_{n;4} \end{bmatrix} \end{bmatrix}$$

Applying (3.1.1), (3.1.2) and (3.2.2), (3.2.10) can be rewritten as

$$\begin{bmatrix} \begin{bmatrix} a^2_{11} & a_{12}a_{21} \\ a_{13}a_{31} & a_{14}a_{41} \end{bmatrix} \begin{bmatrix} A^2_{n;11} \\ A_{n;12}A_{n;21} \end{bmatrix} & \begin{bmatrix} a_{11}a_{12} & a_{12}a_{22} \\ a_{13}a_{32} & a_{14}a_{42} \end{bmatrix} \begin{bmatrix} A_{n;11}A_{n;12} \\ A_{n;12}A_{n;22} \end{bmatrix} \\ \\ \begin{bmatrix} a_{21}a_{11} & a_{22}a_{21} \\ a_{23}a_{31} & a_{24}a_{41} \end{bmatrix} \begin{bmatrix} A_{n;21}A_{n;11} \\ A_{n;22}A_{n;21} \end{bmatrix} & \begin{bmatrix} a_{21}a_{12} & a^2_{22} \\ a_{23}a_{32} & a_{24}a_{42} \end{bmatrix} \begin{bmatrix} A_{n;21}A_{n;12} \\ A^2_{n;22} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} (B_{2;11} \circ \tilde{A}_{2;11})X_{2,n;1} & (B_{2;11} \circ \tilde{A}_{2;12})X_{2,n;2} \\ (B_{2;12} \circ \tilde{A}_{2;11})X_{2,n;3} & (B_{2;12} \circ \tilde{A}_{2;12})X_{2,n;4} \end{bmatrix}. \tag{2.2.18}$$

Therefore, after the entries of $X_{2,n+1;1}$ as in (3.2.10) or (3.2.15) have been permuted, $X_{2,n+1;1}$ can be represented by a 2×2 matrix

$$\hat{X}_{2,n+1;1} \equiv \mathcal{P}(X_{2,n+1;1}) \equiv \begin{bmatrix} X_{2,n+1;1;1} & X_{2,n+1;1;2} \\ X_{2,n+1;1;3} & X_{2,n+1;1;4} \end{bmatrix}, \tag{2.2.19}$$

where

$$\left.\begin{aligned} X_{2,n+1;1;1} &= S_{2;11}X_{2,n;1}, \\ X_{2,n+1;1;2} &= S_{2;12}X_{2,n;2}, \\ X_{2,n+1;1;3} &= S_{2;13}X_{2,n;3}, \\ X_{2,n+1;1;4} &= S_{2;14}X_{2,n;4} \end{aligned}\right\} \tag{2.2.20}$$

and

$$\left.\begin{array}{l} S_{2;11} = B_{2;11} \circ \tilde{A}_{2;11} \equiv C_{2;11}, \\ S_{2;12} = B_{2;11} \circ \tilde{A}_{2;12} \equiv C_{2;12}, \\ S_{2;13} = B_{2;12} \circ \tilde{A}_{2;11} \equiv C_{2;21}, \\ S_{2;14} = B_{2;12} \circ \tilde{A}_{2;12} \equiv C_{2;22}, \end{array}\right\}. \tag{2.2.21}$$

The above derivation indicates that $X_{2,n+1;\alpha}$ can be reduced to $X_{2,n;\beta}$ via multiplication with connecting matrices $C_{2;\alpha\beta}$. This procedure can be extended to introduce the connecting operator $\mathbb{C}_m = [\ C_{m;\alpha\beta}\ ]$, for all $m \geq 2$.

Before $\mathbb{C}_m$ is introduced, three products of matrices are defined as follows.

Definition 2.2. For any two matrices $\mathbb{M} = (M_{ij})$ and $\mathbb{N} = (N_{kl})$, the Kronecker product (tensor product) $\mathbb{M} \otimes \mathbb{N}$ of $\mathbb{M}$ and $\mathbb{N}$ is defined by

$$\mathbb{M} \otimes \mathbb{N} = (M_{ij}\mathbb{N}). \tag{2.2.22}$$

For any $n \geq 1$,

$$\otimes \mathbb{N}^n = \mathbb{N} \otimes \mathbb{N} \otimes \cdots \otimes \mathbb{N},$$

n-times in $\mathbb{N}$.
Next, for any two $m \times m$ matrices

$$\mathbb{P} = (P_{ij}) \textit{ and } \mathbb{Q} = (Q_{ij})$$

where P_{ij} and Q_{ij} are numbers or matrices, the Hadamard product $\mathbb{P} \circ \mathbb{Q}$ is defined by

$$\mathbb{P} \circ \mathbb{Q} = (P_{ij} \cdot Q_{ij}), \tag{2.2.23}$$

where the product $P_{ij} \cdot Q_{ij}$ of P_{ij} and Q_{ij} may be a multiplication between numbers, between numbers and matrices or between matrices whenever it is well-defined.

Finally, product $\hat{\otimes}$ is defined as follows. For any 4×4 matrix

$$\mathbb{M}_2 = \begin{bmatrix} m_{11} & m_{12} & m_{21} & m_{22} \\ m_{13} & m_{14} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{41} & m_{42} \\ m_{33} & m_{34} & m_{43} & m_{44} \end{bmatrix} = \begin{bmatrix} M_{2;1} & M_{2;2} \\ M_{2;3} & M_{2;4} \end{bmatrix} \tag{2.2.24}$$

and any 2×2 matrix

$$\mathbb{N} = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix}, \tag{2.2.25}$$

where m_{ij} are numbers and N_k are numbers or matrices, for $1 \le i, j, k \le 4$, define

(2.2.26)
$$\mathbb{M}_2 \hat{\otimes} \mathbb{N} = \begin{bmatrix} m_{11}N_1 & m_{12}N_2 & m_{21}N_1 & m_{22}N_2 \\ m_{13}N_3 & m_{14}N_4 & m_{23}N_3 & m_{24}N_4 \\ m_{31}N_1 & m_{32}N_2 & m_{41}N_1 & m_{42}N_2 \\ m_{33}N_3 & m_{34}N_4 & m_{43}N_3 & m_{44}N_4 \end{bmatrix}.$$

Furthermore, for $n \ge 1$, the $n+1$ th order of transition matrix of $\mathbb{M}_2$ is defined by

$$\mathbb{M}_{n+1} \equiv \hat{\otimes}\mathbb{M}_2^n = \mathbb{M}_2 \hat{\otimes} \mathbb{M}_2 \hat{\otimes} \cdots \hat{\otimes} \mathbb{M}_2,$$

n-times in $\mathbb{M}_2$. More precisely,

$$\mathbb{M}_{n+1} = \mathbb{M}_2 \hat{\otimes} (\hat{\otimes}\mathbb{M}_2^{n-1}) = \begin{bmatrix} M_{2;1} \circ (\hat{\otimes}\mathbb{M}_2^{n-1}) & M_{2;2} \circ (\hat{\otimes}\mathbb{M}_2^{n-1}) \\ M_{2;3} \circ (\hat{\otimes}\mathbb{M}_2^{n-1}) & M_{2;4} \circ (\hat{\otimes}\mathbb{M}_2^{n-1}) \end{bmatrix}$$

(2.2.27)
$$= \left[\begin{array}{cc|cc} m_{11}M_{n;1} & m_{12}M_{n;2} & m_{21}M_{n;1} & m_{22}M_{n;2} \\ m_{13}M_{n;3} & m_{14}M_{n;4} & m_{23}M_{n;3} & m_{24}M_{n;4} \\ \hline m_{31}M_{n;1} & m_{32}M_{n;2} & m_{41}M_{n;1} & m_{42}M_{n;2} \\ m_{33}M_{n;3} & m_{34}M_{n;4} & m_{43}M_{n;3} & m_{44}M_{n;4} \end{array}\right] = \begin{bmatrix} M_{n+1;1} & M_{n+1;2} \\ M_{n+1;3} & M_{n+1;4} \end{bmatrix},$$

where

$$\mathbb{M}_n = \hat{\otimes}\mathbb{M}_2^{n-1} = \begin{bmatrix} M_{n;1} & M_{n;2} \\ M_{n;3} & M_{n;4} \end{bmatrix}.$$

Here, the following convention is adopted,

$$\hat{\otimes}\mathbb{M}_2^0 = \mathbb{E}_{2\times 2}.$$

Definition 2.3. For $m \ge 2$, define

(2.2.28)
$$\mathbb{C}_m = \begin{bmatrix} C_{m;11} & C_{m;12} & C_{m;13} & C_{m;14} \\ C_{m;21} & C_{m;22} & C_{m;23} & C_{m;24} \\ C_{m;31} & C_{m;32} & C_{m;33} & C_{m;34} \\ C_{m;41} & C_{m;42} & C_{m;43} & C_{m;44} \end{bmatrix} = \begin{bmatrix} S_{m;11} & S_{m;12} & S_{m;21} & S_{m;22} \\ S_{m;13} & S_{m;14} & S_{m;23} & S_{m;24} \\ S_{m;31} & S_{m;32} & S_{m;41} & S_{m;42} \\ S_{m;33} & S_{m;34} & S_{m;43} & S_{m;44} \end{bmatrix},$$

where

(2.2.29)
$$\begin{aligned} C_{m;\alpha\beta} \quad &= \quad \left(\begin{bmatrix} a_{\alpha 1} & a_{\alpha 2} \\ a_{\alpha 3} & a_{\alpha 4} \end{bmatrix} \circ \left(\hat{\otimes} \begin{bmatrix} B_{2;1} & B_{2;2} \\ B_{2;3} & B_{2;4} \end{bmatrix}^{m-2} \right)_{2\times 2} \right)_{2^{m-1}\times 2^{m-1}} \\ &\circ \quad \left(E_{2^{m-2}\times 2^{m-2}} \otimes \left(\begin{bmatrix} a_{1\beta} & a_{2\beta} \\ a_{3\beta} & a_{4\beta} \end{bmatrix} \right) \right)_{2^{m-1}\times 2^{m-1}}. \end{aligned}$$

Similarly, for $\mathbb{B}_2$, define

(2.2.30)

$$\mathbb{U}_m = \begin{bmatrix} U_{m;11} & U_{m;12} & U_{m;13} & U_{m;14} \\ U_{m;21} & U_{m;22} & U_{m;23} & U_{m;24} \\ U_{m;31} & U_{m;32} & U_{m;33} & U_{m;34} \\ U_{m;41} & U_{m;42} & U_{m;43} & U_{m;44} \end{bmatrix} = \begin{bmatrix} W_{m;11} & W_{m;12} & W_{m;21} & W_{m;22} \\ W_{m;13} & W_{m;14} & W_{m;23} & W_{m;24} \\ W_{m;31} & W_{m;32} & W_{m;41} & W_{m;42} \\ W_{m;33} & W_{m;34} & W_{m;43} & W_{m;44} \end{bmatrix},$$

where

(2.2.31)

$$\begin{aligned} U_{m;\alpha\beta} \;=\; & \left(\begin{bmatrix} b_{\alpha_1} & b_{\alpha_2} \\ b_{\alpha_3} & b_{\alpha_4} \end{bmatrix} \circ \left(\hat{\otimes} \begin{bmatrix} A_{2;1} & A_{2;2} \\ A_{2;3} & A_{2;4} \end{bmatrix}^{m-2} \right)_{2\times 2} \right)_{2^{m-1}\times 2^{m-1}} \\ \circ \; & \left(E_{2^{m-2}\times 2^{m-2}} \otimes \left(\begin{bmatrix} b_{1\beta} & b_{2\beta} \\ b_{3\beta} & b_{4\beta} \end{bmatrix} \right) \right)_{2^{m-1}\times 2^{m-1}}. \end{aligned}$$

$\mathbb{S}_m = [S_{m;\alpha\beta}]$ and $\mathbb{W}_m = [W_{m;\alpha\beta}]$.

Now $\mathbb{C}_{m+1}$ can be found from $\mathbb{C}_m$ by a recursive formula, as in (2.1.7).

Theorem 2.4. For any $m \geq 2$ and $1 \leq \alpha, \beta \leq 4$,

$$C_{m+1;\alpha\beta} = \begin{bmatrix} a_{\alpha_1} C_{m;1\beta} & a_{\alpha_2} C_{m;2\beta} \\ a_{\alpha_3} C_{m;3\beta} & a_{\alpha_4} C_{m;4\beta} \end{bmatrix}, \tag{2.2.32}$$

and

$$U_{m+1;\alpha\beta} = \begin{bmatrix} b_{\alpha_1} U_{m;1\beta} & b_{\alpha_2} U_{m;2\beta} \\ b_{\alpha_3} U_{m;3\beta} & b_{\alpha_4} U_{m;4\beta} \end{bmatrix}. \tag{2.2.33}$$

Proof. By (3.2.43),

$$\hat{\otimes}\mathbb{B}_2^{m-1} = \mathbb{B}_2 \hat{\otimes} (\hat{\otimes}\mathbb{B}_2^{m-2}) = \begin{bmatrix} B_{2;1} \circ (\hat{\otimes}\mathbb{B}_2^{m-2}) & B_{2;2} \circ (\hat{\otimes}\mathbb{B}_2^{m-2}) \\ B_{2;3} \circ (\hat{\otimes}\mathbb{B}_2^{m-2}) & B_{2;4} \circ (\hat{\otimes}\mathbb{B}_2^{m-2}) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} & C_{m+1;\alpha\beta} = (B_{2;\alpha} \circ (\hat{\otimes}\mathbb{B}_2^{m-1})) \circ (E_{2^{m-1}\times 2^{m-1}} \otimes \tilde{A}_{2;\beta}) \\ & = \begin{bmatrix} a_{\alpha 1}(B_{2;1} \circ \hat{\otimes}\mathbb{B}_2^{m-2}) & a_{\alpha 2}(B_{2;2} \circ \hat{\otimes}\mathbb{B}_2^{m-2}) \\ a_{\alpha 3}(B_{2;3} \circ \hat{\otimes}\mathbb{B}_2^{m-2}) & a_{\alpha 4}(B_{2;4} \circ \hat{\otimes}\mathbb{B}_2^{m-2}) \end{bmatrix} \circ (E_{2^{m-1}\times 2^{m-1}} \otimes \tilde{A}_{2;\beta}) \\ & = \begin{bmatrix} a_{\alpha 1}[(B_{2;1} \circ \hat{\otimes}\mathbb{B}_2^{m-2}) \circ (E_{2^{m-2}\times 2^{m-2}} \otimes \tilde{A}_{2;\beta})] & a_{\alpha 2}[(B_{2;2} \circ \hat{\otimes}\mathbb{B}_2^{m-2}) \circ (E_{2^{m-2}\times 2^{m-2}} \otimes \tilde{A}_{2;\beta}) \\ a_{\alpha 3}[(B_{2;3} \circ \hat{\otimes}\mathbb{B}_2^{m-2}) \circ (E_{2^{m-2}\times 2^{m-2}} \otimes \tilde{A}_{2;\beta})] & a_{\alpha 4}[(B_{2;4} \circ \hat{\otimes}\mathbb{B}_2^{m-2}) \circ (E_{2^{m-2}\times 2^{m-2}} \otimes \tilde{A}_{2;\beta})] \end{bmatrix} \\ & = \begin{bmatrix} a_{\alpha 1} C_{m;1\beta} & a_{\alpha 2} C_{m;2\beta} \\ a_{\alpha 3} C_{m;3\beta} & a_{\alpha 4} C_{m;4\beta} \end{bmatrix}. \end{aligned}$$

A similar result also holds for $U_{m;\alpha\beta}$; the details are omitted here. The proof is complete. $\square$

Notably, (3.2.51) implies $\mathbb{C}_{m;ij}$ is $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_m i_{m+1}}$ with $i_1 = i$ and $i_{m+1} = j$. $\mathbb{C}_{m;ij}$ consist of all words(or paths) of length m starting from i and ending at j. Indeed, the entries of $\mathbb{C}_m$ and $\mathbb{B}_{m+1}$ are the same. However, the arrangements are different. $\mathbb{C}_m$ can also be used to study the primitivity of $\mathbb{A}_n$, $n \geq 2$, as in [6].

That the recursive formula (2.1.24) holds remains to be shown. Indeed, in (2.2.6) substituting n for $n+1$ and using (2.1.7),

$$
\begin{aligned}
& A^{(k)}_{m,n+1;\alpha} \\
&= A_{n+1;j_1 j_2} A_{n+1;j_2 j_3} \cdots A_{n+1,j_m j_{m+1}} \\
&= \prod_{i=1}^{m} \begin{bmatrix} b_{\alpha_i 1} A_{n;11} & b_{\alpha_i 2} A_{n;12} \\ b_{\alpha_i 3} A_{n;21} & b_{\alpha_i 4} A_{n;22} \end{bmatrix}
\end{aligned} \tag{2.2.34}
$$

where $\alpha_i = \alpha(j_i, j_{i+1})$, for $1 \leq i \leq m$. After m matrix multiplications are executed in (3.2.49),

$$
A^{(k)}_{m,n+1;\alpha} = \begin{bmatrix} A^{(k)}_{m,n+1;\alpha;1} & A^{(k)}_{m,n+1;\alpha;2} \\ A^{(k)}_{m,n+1;\alpha;3} & A^{(k)}_{m,n+1;\alpha;4} \end{bmatrix} \tag{2.2.35}
$$

where

$$
A^{(k)}_{m,n+1;\alpha;\beta} = \sum_{l=1}^{2^{m-1}} K(m;\alpha,\beta;k,l) A^{(l)}_{m,n;\beta} \tag{2.2.36}
$$

is a linear combination of $A^{(l)}_{m,n;\beta}$ with the coefficients $K(m;\alpha,\beta;k,l)$ which are products of $b_{\alpha_l j}, 1 \leq l \leq m$. $K(m;\alpha,\beta;k,l)$ must be studied in more details.

Note that

$$
\mathbb{A}^m_{n+1} = \begin{bmatrix} A_{m,n+1;1} & A_{m,n+1;2} \\ A_{m,n+1;3} & A_{m,n+1;4} \end{bmatrix} \tag{2.2.37}
$$

$$
= \begin{bmatrix} \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;1} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;2} \\ \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;3} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;4} \end{bmatrix}
$$

$$= \begin{bmatrix} \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;1;1} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;1;2} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;2;1} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;2;2} \\ \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;1;3} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;1;4} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;2;3} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;2;4} \\ \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;3;1} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;3;2} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;4;1} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;4;2} \\ \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;3;3} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;3;4} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;4;3} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;4;4} \end{bmatrix}$$

Now, $X_{m,n+1;\alpha;\beta}$ is defined as

$$X_{m,n+1;\alpha;\beta} = (A^{(k)}_{m,n+1;\alpha;\beta})^t. \tag{2.2.38}$$

As in (3.2.10), the entries of $X_{m,n+1;\alpha}$ are rearranged into a new matrix

$$\hat{X}_{m,n+1;\alpha} \equiv \mathcal{P}(X_{m,n+1;\alpha}) \equiv \begin{bmatrix} X_{m,n+1;\alpha;1} & X_{m,n+1;\alpha;2} \\ X_{m,n+1;\alpha;3} & X_{m,n+1;\alpha;4} \end{bmatrix}. \tag{2.2.39}$$

From (2.2.36) and (3.2.54),

$$X_{m,n+1;\alpha;\beta} = \mathbb{K}(m;\alpha,\beta) X_{m,n;\beta} \tag{2.2.40}$$

where

$$\mathbb{K}(m;\alpha,\beta) = (K(m;\alpha,\beta;k,l)),\ 1 \leq k,l \leq 2^{m-1},$$

is a $2^{m-1} \times 2^{m-1}$ matrix. Now, $\mathbb{K}(m;\alpha,\beta) = S_{m;\alpha\beta}$ must be shown as follows.

Theorem 2.5. For any $m \geq 2$ and $n \geq 2$, let $S_{m;\alpha\beta}$ be given as in (3.2.44) and (3.2.45). Then,

$$\mathbb{K}(m;\alpha,\beta) = S_{m;\alpha\beta},$$

i.e.,

$$X_{m,n+1;\alpha;\beta} = S_{m;\alpha\beta} X_{m,n;\beta}, \tag{2.2.41}$$

or equivalently, the recursive formula (2.1.24) holds. That is,

$$A^{(k)}_{m,n+1;\alpha} = \begin{bmatrix} \sum_{l=1}^{2^{m-1}} (S_{m;\alpha 1})_{kl} A^{(l)}_{m,n;1} & \sum_{l=1}^{2^{m-1}} (S_{m;\alpha 2})_{kl} A^{(l)}_{m,n;2} \\ \sum_{l=1}^{2^{m-1}} (S_{m;\alpha 3})_{kl} A^{(l)}_{m,n;3} & \sum_{l=1}^{2^{m-1}} (S_{m;\alpha 4})_{kl} A^{(l)}_{m,n;4} \end{bmatrix}. \tag{2.2.42}$$

Moreover, for $n = 1$,

$$A^{(k)}_{m,2;\alpha} = \begin{bmatrix} \sum_{l=1}^{2^{m-1}} (S_{m;\alpha 1})_{kl} & \sum_{l=1}^{2^{m-1}} (S_{m;\alpha 2})_{kl} \\ \sum_{l=1}^{2^{m-1}} (S_{m;\alpha 3})_{kl} & \sum_{l=1}^{2^{m-1}} (S_{m;\alpha 4})_{kl} \end{bmatrix} \tag{2.2.43}$$

for any $1 \leq k \leq 2^{m-1}$ and $\alpha \in \{1,2,3,4\}$.

Proof. The result is proven by the induction on m.

When $m = 2$, and $\alpha = 1$, (3.2.59) was proven as in Example 2.1. The case with $\alpha = 2$, 3 and 4 can also be proved analogously; the details are omitted.

Now, (3.2.59) ia assumed to hold for m; the goal is to show that it also holds for $m+1$. Since

$$\mathbb{A}_{n+1}^{m+1} = \mathbb{A}_{n+1} \cdot \mathbb{A}_{n+1}^{m} = \begin{bmatrix} A_{n+1;1} & A_{n+1;2} \\ A_{n+1;3} & A_{n+1;4} \end{bmatrix} \begin{bmatrix} A_{m,n+1,1} & A_{m,n+1;2} \\ A_{m,n+1,3} & A_{m,n+1;4} \end{bmatrix},$$

(2.2.11) implies

$$X_{m+1,n+1;1} = \begin{bmatrix} A_{n+1;1}X_{m,n+1;1} \\ A_{n+1;2}X_{m,n+1;3} \end{bmatrix},\ X_{m+1,n+1;2} = \begin{bmatrix} A_{n+1;1}X_{m,n+1;2} \\ A_{n+1;2}X_{m,n+1;4} \end{bmatrix},$$

$$X_{m+1,n+1;3} = \begin{bmatrix} A_{n+1;3}X_{m,n+1;1} \\ A_{n+1;4}X_{m,n+1;3} \end{bmatrix},\ \text{and } X_{m+1,n+1;4} = \begin{bmatrix} A_{n+1;3}X_{m,n+1;2} \\ A_{n+1;4}X_{m,n+1;4} \end{bmatrix}.$$

For $\alpha = 1$, by induction on m,

$$(A_{n+1;1}\mathcal{P}(X_{m,n+1;1}),\ A_{n+1;2}\mathcal{P}(X_{m,n+1;3}))^t$$

$$= \begin{bmatrix} \begin{bmatrix} b_{11}A_{n;1} & b_{12}A_{n;2} \\ b_{13}A_{n;3} & b_{14}A_{n;4} \end{bmatrix} & \begin{bmatrix} S_{m;11}X_{m,n;1} & S_{m;12}X_{m,n;2} \\ S_{m;13}X_{m,n;3} & S_{m;14}X_{m,n;4} \end{bmatrix} \\ \\ \begin{bmatrix} b_{21}A_{n;1} & b_{22}A_{n;2} \\ b_{23}A_{n;3} & b_{24}A_{n;4} \end{bmatrix} & \begin{bmatrix} S_{m;31}X_{m,n;1} & S_{m;32}X_{m,n;2} \\ S_{m;33}X_{m,n;3} & S_{m;34}X_{m,n;4} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} b_{11}S_{m;11}A_{n;1}X_{m,n;1} + b_{12}S_{m;13}A_{n;2}X_{m,n;3} & b_{11}S_{m;12}A_{n;1}X_{m,n;2} + b_{12}S_{m;14}A_{n;2}X_{m,n;4} \\ b_{13}S_{m;11}A_{n;3}X_{m,n;1} + b_{14}S_{m;13}A_{n;4}X_{m,n;3} & b_{13}S_{m;12}A_{n;3}X_{m,n;2} + b_{14}S_{m;14}A_{n;4}X_{m,n;4} \end{bmatrix} \\ \\ \begin{bmatrix} b_{21}S_{m;31}A_{n;1}X_{m,n;1} + b_{22}S_{m;33}A_{n;2}X_{m,n;3} & b_{21}S_{m;32}A_{n;1}X_{m,n;2} + b_{22}S_{m;34}A_{n;2}X_{m,n;4} \\ b_{23}S_{m;31}A_{n;3}X_{m,n;1} + b_{24}S_{m;33}A_{n;4}X_{m,n;3} & b_{23}S_{m;32}A_{n;3}X_{m,n;2} + b_{24}S_{m;34}A_{n;4}X_{m,n;4} \end{bmatrix} \end{bmatrix}$$

Hence $X_{m+1,n+1;1}$ can be represented by a matrix

$$\hat{X}_{m+1,n+1;1} \equiv \mathcal{P}(X_{m+1,n+1;1}) \equiv \begin{bmatrix} X_{m+1,n+1;1,1} & X_{m+1,n+1;1,2} \\ X_{m+1,n+1;1,3} & X_{m+1,n+1;1,4} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} b_{11}S_{m;11} & b_{12}S_{m;13} \\ b_{21}S_{m;31} & b_{22}S_{m;33} \end{bmatrix} \begin{bmatrix} A_{n;1}X_{m,n;1} \\ A_{n;2}X_{m,n;3} \end{bmatrix} & \begin{bmatrix} b_{11}S_{m;12} & b_{12}S_{m;14} \\ b_{21}S_{m;32} & b_{22}S_{m;34} \end{bmatrix} \begin{bmatrix} A_{n;1}X_{m,n;2} \\ A_{n;2}X_{m,n;4} \end{bmatrix} \\ \\ \begin{bmatrix} b_{13}S_{m;11} & b_{14}S_{m;13} \\ b_{23}S_{m;31} & b_{24}S_{m;33} \end{bmatrix} \begin{bmatrix} A_{n;3}X_{m,n;1} \\ A_{n;4}X_{m,n;3} \end{bmatrix} & \begin{bmatrix} b_{13}S_{m;12} & b_{14}S_{m;14} \\ b_{23}S_{m;32} & b_{24}S_{m;34} \end{bmatrix} \begin{bmatrix} A_{n;3}X_{m,n;2} \\ A_{n;4}X_{m,n;4} \end{bmatrix} \end{bmatrix}$$

Once again, (3.1.1), (3.1.2) and (3.2.2) can be used to recast the matrix $\hat{X}_{m+1,n+1;1}$ as

$$\begin{bmatrix} \begin{bmatrix} a_{11}C_{m;11} & a_{12}C_{m;21} \\ a_{13}C_{m;31} & a_{14}C_{m;41} \end{bmatrix} X_{m+1,n;1} & \begin{bmatrix} a_{11}C_{m;12} & a_{12}C_{m;22} \\ a_{13}C_{m;32} & a_{14}C_{m;42} \end{bmatrix} X_{m+1,n;2} \\ \\ \begin{bmatrix} a_{21}C_{m;11} & a_{22}C_{m;21} \\ a_{23}C_{m;31} & a_{24}C_{m;41} \end{bmatrix} X_{m+1,n;3} & \begin{bmatrix} a_{21}C_{m;12} & a_{22}C_{m;22} \\ a_{23}C_{m;32} & a_{24}C_{m;42} \end{bmatrix} X_{m+1,n;4} \end{bmatrix}$$

According to Theorem 2.4, the above matrix becomes

$$= \begin{bmatrix} C_{m+1;11}X_{m+1,n;1} & C_{m+1;12}X_{m+1,n;2} \\ C_{m+1;21}X_{m+1,n;3} & C_{m+1;22}X_{m+1,n;4} \end{bmatrix} = \begin{bmatrix} S_{m+1;11}X_{m+1,n;1} & S_{m+1;12}X_{m+1,n;2} \\ S_{m+1;13}X_{m+1,n;3} & S_{m+1;14}X_{m+1,n;4} \end{bmatrix}.$$

The cases with $\alpha = 2, 3$ and 4 can also be considered analogously (3.2.59) follows.

Next, (3.2.60) follows easily from (2.2.35), (2.2.36) and (3.2.59).

Equation (3.2.61) remains to be shown. If the 2×2 matrix

$$\mathbb{A}_1 \equiv \begin{bmatrix} A_{1;11} & A_{1;12} \\ A_{1;21} & A_{1;22} \end{bmatrix} \equiv \begin{bmatrix} A_{1;1} & A_{1;2} \\ A_{1;3} & A_{1;4} \end{bmatrix} \equiv \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{2.2.44}$$

is introduced, then the previous argument also hold for $n = 1$. Hence, (3.2.61) holds. The proof is complete. □

For any positive integer $p \geq 2$, applying Theorem 2.5 p times permits the elementary patterns of $\mathbb{A}^m_{n+p}$ to be expressed as the product of a sequence of $S_{m;\beta_i\beta_{i+1}}$ and the elementary patterns in $\mathbb{A}^m_n$. The elementary pattern in $\mathbb{A}^m_{n+p}$ is first studied.

For any $p \geq 2$ and $1 \leq q \leq p-1$, define

$$A^{(k)}_{m,n+p;\alpha;\beta_1;\beta_2;\cdots;\beta_q} = \begin{bmatrix} A^{(k)}_{m,n+p;\alpha;\beta_1;\beta_2;\cdots;\beta_q;1} & A^{(k)}_{m,n+p;\alpha;\beta_1;\beta_2;\cdots;\beta_q;2} \\ A^{(k)}_{m,n+p;\alpha;\beta_1;\beta_2;\cdots;\beta_q;3} & A^{(k)}_{m,n+p;\alpha;\beta_1;\beta_2;\cdots;\beta_q;4} \end{bmatrix}. \tag{2.2.45}$$

Then

$$A^{(k)}_{m,n+p;\alpha;\beta_1;\beta_2;\cdots;\beta_p} = \sum_{l_1=1}^{2^{m-1}} \cdots \sum_{l_p=1}^{2^{m-1}} \Big(\prod_{i=1}^{p} K(m;\beta_{i-1},\beta_i;l_{i-1},l_i)\Big) A^{(l_p)}_{m,n;\beta_p}, \tag{2.2.46}$$

where $\beta_0 = \alpha$ and $l_0 = k$ can be easily verified. Therefore, for any $p \geq 1$, a generalization for (3.2.53) can be found for $\mathbb{A}^m_{n+p}$ as a $2^{p+1} \times 2^{p+1}$ matrix

$$\mathbb{A}^m_{n+p} = \left[A_{m,n+p;\alpha;\beta_1;\beta_2\cdots;\beta_p}\right] \tag{2.2.47}$$

where

$$A_{m,n+p;\alpha;\beta_1;\beta_2\cdots;\beta_p} = \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n;\alpha;\beta_1;\beta_2\cdots;\beta_p}. \tag{2.2.48}$$

In particular, if $\alpha;\beta_1,\beta_2\cdots,\beta_p \in \{1,4\}$, then $A_{m,n+p;\alpha;\beta_1;\beta_2\cdots;\beta_p}$ lies on the diagonal of $\mathbb{A}^m_{n+p}$ in (2.2.47).

Now, define

$$X_{m,n+p;\alpha;\beta_1;\beta_2;\cdots;\beta_p} = (A^{(k)}_{m,n+p;\alpha;\beta_1;\beta_2;\cdots;\beta_p})^t. \tag{2.2.49}$$

Therefore, Theorem 2.5 can be generalized to

Theorem 2.6. For any $m \geq 2,\ n \geq 2$ and $p \geq 1$,

$$X_{m,n+p;\alpha;\beta_1;\beta_2\cdots;\beta_p} = S_{m;\alpha\beta_1} S_{m;\beta_1\beta_2} \cdots S_{m;\beta_{p-1}\beta_p} X_{m,n;\beta_p} \tag{2.2.50}$$

where $\alpha, \beta_i \in \{1,2,3,4\}$ and $1 \leq i \leq p$.

Proof. From (2.2.46), (3.2.58) and (3.2.60),

$$\begin{aligned}
A^{(k)}_{m,n+p;\alpha;\beta_1;\beta_2;\cdots;\beta_p} &= \sum_{l_1=1}^{2^{m-1}} \cdots \sum_{l_p=1}^{2^{m-1}} \left(\prod_{i=1}^{p} K(m;\beta_{i-1},\beta_i;l_{i-1},l_i)\right) A^{(l_p)}_{m,n;\beta_p} \\
&= \sum_{l_1=1}^{2^{m-1}} \cdots \sum_{l_p=1}^{2^{m-1}} \left(\prod_{i=1}^{p} (S_{m;\beta_{i-1}\beta_i})_{l_{i-1}l_i}\right) A^{(l_p)}_{m,n;\beta_p} \\
&= \sum_{l_1=1}^{2^{m-1}} \cdots \sum_{l_p=1}^{2^{m-1}} (S_{m;\beta_0\beta_1})_{l_0 l_1} (S_{m;\beta_1\beta_2})_{l_1 l_2} \cdots (S_{m;\beta_{p-1}\beta_p})_{l_{p-1} l_p} A^{(l_p)}_{m,n;\beta_p} \\
&= \sum_{l_p=1}^{2^{m-1}} (S_{m;\beta_0\beta_1} S_{m;\beta_1\beta_2} \cdots S_{m;\beta_{p-1}\beta_p})_{l_0 l_p} A^{(l_p)}_{m,n;\beta_p} \\
&= \sum_{l_p=1}^{2^{m-1}} (S_{m;\alpha\beta_1} S_{m;\beta_1\beta_2} \cdots S_{m;\beta_{p-1}\beta_p})_{k l_p} A^{(l_p)}_{m,n;\beta_p}
\end{aligned}$$

is derived. By (2.2.49), then

$$\begin{aligned}
X_{m,n+p;\alpha;\beta_1;\beta_2;\cdots;\beta_p} &= (A^{(k)}_{m,n+p;\alpha;\beta_1;\beta_2;\cdots;\beta_p})^t \\
&= \left(\sum_{l_p=1}^{2^{m-1}} (S_{m;\alpha\beta_1} S_{m;\beta_1\beta_2} \cdots S_{m;\beta_{p-1}\beta_p})_{k l_p} A^{(l_p)}_{m,n;\beta_p}\right)^t \\
&= S_{m;\alpha\beta_1} S_{m;\beta_1\beta_2} \cdots S_{m;\beta_{p-1}\beta_p} X_{m,n;\beta_p}.
\end{aligned}$$

The proof is complete. □

2.2.2 Lower bound of entropy

In this subsection, the connecting operator $\mathbb{C}_m$ is employed to estimate the lower bound of entropy, and in particular, to verify the positivity of entropy.

First, recall some properties of $\Gamma_{m,n}$ and spatial entropy.

$\Gamma_{m,n}$ satisfies the subadditivity in m and n:

$$\Gamma_{m_1+m_2,n} \leq \Gamma_{m_1,n}\Gamma_{m_2,n}, \tag{2.2.51}$$

and

$$\Gamma_{m,n_1+n_2} \leq \Gamma_{m,n_1}\Gamma_{m,n_2}, \tag{2.2.52}$$

or equivalently,

$$|\mathbb{A}_n^{m_1+m_2}| \leq |\mathbb{A}_n^{m_1}||\mathbb{A}_n^{m_2}| \tag{2.2.53}$$

and

$$|\mathbb{A}_{n_1+n_2}^{m}| \leq |\mathbb{A}_{n_1}^{m}||\mathbb{A}_{n_2}^{m}|, \tag{2.2.54}$$

for positive integers m, n, m_1, n_1, m_2 and n_2. Here

$$\mathbb{A}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{2.2.55}$$

is applied.

The subadditivity property implies

$$\limsup_{m,n\to\infty} \frac{1}{mn} \log|\mathbb{A}_n^m| \leq \frac{1}{pq} \log|\mathbb{A}_q^{p-1}| \tag{2.2.56}$$

for any p and $q \geq 2$. Therefore,

$$h(\mathbb{A}_2) = \lim_{m,n\to\infty} \frac{1}{mn} \log|\mathbb{A}_n^m|$$

exists, and equals

$$\inf_{p,q\geq 2} \frac{1}{pq} \log|\mathbb{A}_q^{p-1}|. \tag{2.2.57}$$

In particular, $h(\mathbb{A}_2)$ has an upper bound

$$h(\mathbb{A}_2) \leq \frac{1}{pq} \log|\mathbb{A}_q^{p-1}| \tag{2.2.58}$$

for any p and $q \geq 2$.

Similarly, when $\mathbb{A}_2$ is horizontal (or vertical) transition matrix for any $m \geq 1$ and $q \geq 2$,

$$\limsup_{n\to\infty} \frac{1}{n} \log |\mathbb{A}_n^m| \leq \frac{1}{q} \log |\mathbb{A}_q^m|. \tag{2.2.59}$$

Hence, the spatial entropy is $h_m(\mathbb{A}_2)$ on an infinite lattice $\mathbb{Z}_{m+1\times\infty}$ (or $\mathbb{Z}_{\infty\times m+1}$) and

$$h_m(\mathbb{A}_2) \equiv \lim_{n\to\infty} \frac{1}{n} \log |\mathbb{A}_n^m| = \inf_{q\geq 2} \frac{1}{q} \log |\mathbb{A}_q^m|. \tag{2.2.60}$$

For the proof of the above results, see [15].

Furthermore, by Perron-Frobenius theorem,

$$\lim_{m\to\infty} \frac{1}{m} \log |\mathbb{A}_n^m| = \log \rho(\mathbb{A}_n). \tag{2.2.61}$$

Therefore, for any $n \geq 2$

$$h(\mathbb{A}_2) \leq \frac{1}{n} \log \rho(\mathbb{A}_n). \tag{2.2.62}$$

For a proof of (2.2.61), see [4], [30].

The following notation is adopted.

Definition 2.7. Let $X = (X_1, \cdots, X_M)^t$, where X_k are $N \times N$ matrices. Define the summation of X_k by

$$|X| = \sum_{k=1}^{N} X_k. \tag{2.2.63}$$

If $\mathbb{M} = [M_{ij}]$ is a $M \times M$ matrix, then

$$|\mathbb{M}X| = \sum_{i=1}^{M} \sum_{j=1}^{M} M_{ij} X_j. \tag{2.2.64}$$

Note that, (2.2.63) implies

$$|X_{m,n;\alpha}| = \sum_{k=1}^{2^{m-1}} A_{m,n;\alpha}^{(k)} = A_{m,n;\alpha}. \tag{2.2.65}$$

As usual, the set of all matrices with the same order can be partially ordered.

Definition 2.8. Let $\mathbb{M} = [M_{ij}]$ and $\mathbb{N} = [N_{ij}]$ be two $M \times M$ matrices, $\mathbb{M} \geq \mathbb{N}$ if $M_{ij} \geq N_{ij}$ for all $1 \leq i, j \leq M$.

Notably, if $\mathbb{A}_2 \geq \mathbb{A}'_2$ then $\mathbb{A}_n \geq \mathbb{A}'_n$ for all $n \geq 2$. Therefore, $h(\mathbb{A}_2) \geq h(\mathbb{A}'_2)$. Hence, the spatial entropy as a function of $\mathbb{A}_2$ is monotonic with respect to the partial order $\geq$.

Definition 2.9. A $K+1$ multiple index

$$\mathcal{B}_K \equiv (\beta_1\beta_2 \cdots \beta_K\beta_{K+1}) \tag{2.2.66}$$

is called a (periodic) cycle if

$$\beta_{K+1} = \beta_1. \tag{2.2.67}$$

It is called a diagonal cycle if (2.2.67) holds and

$$\beta_k \in \{1, 4\} \tag{2.2.68}$$

for each $1 \leq k \leq K+1$.

For a diagonal cycle (2.2.66), denote

$$\bar{\beta}_K = \beta_1; \beta_2; \cdots ; \beta_K \tag{2.2.69}$$

and

$$\bar{\beta}_K^n = \bar{\beta}_K; \bar{\beta}_K; \cdots ; \bar{\beta}_K. \text{ (n times)} \tag{2.2.70}$$

First, prove the following Lemma.

Lemma 2.10. Let $m \geq 2$, $K \geq 1$, $\mathcal{B}_K$ be a diagonal cycle. Then, for any $n \geq 1$,

$$\rho(\mathbb{A}^m_{nK+2}) \geq \rho(|(S_{m;\beta_1\beta_2}S_{m;\beta_2\beta_3} \cdots S_{m;\beta_K\beta_{K+1}})^n X_{m,2;\beta_1}|) \tag{2.2.71}$$

Proof. Since $\mathcal{B}_K$ is a periodic cycle, Theorem 2.6 implies

$$X_{m,nK+2;\bar{\beta}^n_K} = (S_{m;\beta_1\beta_2}S_{m;\beta_2\beta_3} \cdots S_{m;\beta_K\beta_{K+1}})^n X_{m,2;\beta_1}. \tag{2.2.72}$$

Furthermore $\mathcal{B}_K$ is diagonal, and $|X_{m,nK+2;\bar{\beta}^n_k}| = A_{m,nK+2;\bar{\beta}^n_k}$ lies on the diagonal part as in (2.2.47) with $n + p = nK + 2$, therefore

$$\rho(\mathbb{A}^m_{nK+2}) \geq \rho(|X_{m,nK+2;\bar{\beta}^n_K}|). \tag{2.2.73}$$

Therefore, (2.2.71) follows from (2.2.72) and (2.2.73).

The proof is complete. □

The following lemma is valuable in studying maximum eigenvalue of $(S_{m;\beta_1\beta_2}\cdots S_{m;\beta_K\beta_{K+1}})^n X_{m,2;\beta_1}$ in (2.2.71).

Lemma 2.11. For any $m \geq 2,\ 1 \leq k \leq 2^{m-1}$ and $\alpha \in \{1,4\}$, if

$$tr(A^{(k)}_{m,2;\alpha}) = 0, \tag{2.2.74}$$

then for all $1 \leq l \leq 2^{m-1}$,

$$(S_{m,\alpha 1})_{kl} = 0 \text{ and } (S_{m;\alpha 4})_{kl} = 0, \tag{2.2.75}$$

i.e., the k-th rows of matrices $S_{m;\alpha 1}$ and $S_{m;\alpha 4}$ are zeros. Furthermore, for any diagonal cycle $\mathcal{B}_K$, let $U = (u_1, u_2, \cdots, u_{2^{m-1}})$ be an eigenvector of $S_{m;\beta_1\beta_2}S_{m;\beta_2\beta_3}\cdots S_{m;\beta_K\beta_1}$, if $u_k \neq 0$ for some $1 \leq k \leq 2^{m-1}$, then

$$tr(A^{(k)}_{m,2;\alpha}) > 0. \tag{2.2.76}$$

Proof. Since $A^{(k)}_{m,2;\alpha}$ can be expressed as in (3.2.61). Therefore, $tr(A^{(k)}_{m,2;\alpha}) = 0$ if and only if (2.2.75) holds for all $1 \leq l \leq 2^{m-1}$. The second part of the lemma follows easily from the first part.

The proof is complete. □

By Lemma 2.10 and Lemma 2.11, the lower bound of entropy can be obtained as follows.

Theorem 2.12. Let $\beta_1\beta_2\cdots\beta_K\beta_1$ be a diagonal cycle. Then for any $m \geq 2$,

$$h(\mathbb{A}_2) \geq \frac{1}{mK}\log\rho(S_{m;\beta_1\beta_2}S_{m;\beta_2\beta_3}\cdots S_{m;\beta_K\beta_1}). \tag{2.2.77}$$

and

$$h(\mathbb{A}_2) \geq \frac{1}{mK}\log\rho(W_{m;\beta_1\beta_2}W_{m;\beta_2\beta_3}\cdots W_{m;\beta_K\beta_1}). \tag{2.2.78}$$

In particular, if a diagonal cycle $\beta_1\beta_2\cdots\beta_K\beta_1$ exists and $m \geq 2$ such that

$$\rho(S_{m;\beta_1\beta_2}S_{m;\beta_2\beta_3}\cdots S_{m;\beta_K\beta_1}) > 1,$$

or

$$\rho(W_{m;\beta_1\beta_2}W_{m;\beta_2\beta_3}\cdots W_{m;\beta_K\beta_1}) > 1$$

then $h(\mathbb{A}_2) > 0$.

Proof. First, show that

$$h(\mathbb{A}_2) \geq \frac{1}{mK} \limsup_{n\to\infty} (\log \rho(|(S_{m;\beta_1\beta_2} S_{m;\beta_2\beta_3} \cdots S_{m;\beta_K\beta_1})^n X_{m,2;\beta_1}|). \tag{2.2.79}$$

Indeed, from (2.1.11) and (2.2.71),

$$\begin{aligned} h(\mathbb{A}_2) &= \lim_{n\to\infty} \frac{1}{nK+2} \log \rho(\mathbb{A}_{nK+2}) \\ &= \lim_{n\to\infty} \frac{1}{m(nK+2)} \log \rho(\mathbb{A}^m_{nK+2}) \\ &\geq \frac{1}{mK} \limsup_{n\to\infty} \frac{1}{n} (\log \rho(|(S_{m;\beta_1\beta_2} \cdots S_{m;\beta_K\beta_1})^n X_{m,2;\beta_1}|)). \end{aligned}$$

Now, the following remains to be shown

$$\limsup_{n\to\infty} \frac{1}{n} (\log \rho(|(S_{m;\beta_1\beta_2} \cdots S_{m;\beta_K\beta_1})^n X_{m,2;\beta_1}|) = \log \rho(S_{m;\beta_1\beta_2} \cdots S_{m;\beta_K\beta_1}). \tag{2.2.80}$$

Since $X_{m,2;\beta_1} = (A^{(k)}_{m,2;\beta_1})^t$, if $tr(A^{(k)}_{m,2;\beta_1}) = 0$ then Lemma 2.11 implies the k-th row of $S_{m;\beta_1\beta_2}$ is zero which implies that the k-th row of $(S_{m;\beta_1\beta_2} \cdots S_{m;\beta_K\beta_1})^n$ is also zero for any $n \geq 1$.

If $tr(A^{(k)}_{m,2;\beta_1}) = 0$ for all $1 \leq k \leq 2^{m-1}$, then $S_{m;\beta_1\beta_2} \equiv 0$. (2.2.80) holds trivially.

Now, assume that $1 \leq k' \leq 2^{m-1}$ exists such that $tr(A^{(k')}_{m,2;\beta_1}) > 0$. Define

$$\hat{X} = (A^{(k')}_{m,2;\beta_1})^t = (\hat{X}_1, \cdots, \hat{X}_M), \tag{2.2.81}$$

where $tr(A^{(k')}_{m,2;\beta_1}) > 0$ for $1 \leq k' \leq M \leq 2^{m-1}$. Then $\rho(\hat{X}_j) > 0$ for $1 \leq j \leq M$.

Let $\mathbb{M}$ be the $M \times M$ sub-matrix of $S_{m;\beta_1\beta_2} \cdots S_{m;\beta_K\beta_1}$ from which the k-th row and k-th column have been removed whenever $tr(A^{(k)}_{m,2;\beta_1}) = 0$ for $1 \leq k \leq 2^{m-1}$.

Clearly,

$$|(S_{m;\beta_1\beta_2} \cdots S_{m;\beta_K\beta_1})^n X_{m,2;\beta_1}| = |\mathbb{M}^n \hat{X}|, \tag{2.2.82}$$

and

$$\rho(S_{m;\beta_1\beta_2} \cdots S_{m;\beta_K\beta_1}) = \rho(\mathbb{M}). \tag{2.2.83}$$

The proof of (2.2.80) comprise three steps, according to

(i) $\mathbb{M}$ is primitive,

(ii) $\mathbb{M}$ is irreducible, and

(iii) $\mathbb{M}$ is reducible.

(i) $\mathbb{M}$ is primitive. Then by Perron-Frobenius Theorem the maximum eigenvalue $\rho(\mathbb{M})$ of $\mathbb{M}$ is unique with maximum modulus, i.e.

$$\rho(\mathbb{M}) = \lambda_1 > |\lambda_j|, \tag{2.2.84}$$

for all $2 \le j \le M$, where λ_j are eigenvalues of $\mathbb{M}$. Moreover, a positive eigenvector $\mathbf{v}_1 = (v_1, v_2, \cdots, v_M)^t$ is associated with λ_1 [26], [27]. Furthermore, Jordan canonical form theorem states that a non-singular matrix $\mathbb{P} = [P_{ij}]_{M\times M}$ exists, such that the real Jordan canonical form of $\mathbb{M}$ is

$$\hat{\mathbb{M}} \equiv \mathbb{P}\mathbb{M}\mathbb{P}^{-1} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & J_{n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & J_{n_q} \end{bmatrix}, \tag{2.2.85}$$

where J_{n_k}, $2 \le k \le q$ are real Jordan blocks and the associated eigenvalue λ_k of J_{n_k} satisfies (2.2.84). Moreover, the positivity of eigenvector $\mathbf{v}_1$ implies that $\mathbb{P}$ can be chosen such that

$$\sum_{i=1}^{M} P_{ij} = 1 \tag{2.2.86}$$

and

$$P_{1j} > 0 \tag{2.2.87}$$

for all $1 \le j \le M$. Therefore, by (2.2.86)

$$\begin{aligned} &|\mathbb{M}^n \hat{X}| = |\mathbb{P}\mathbb{M}^n \hat{X}| = |\mathbb{P}\mathbb{M}^n\mathbb{P}^{-1}\mathbb{P}\hat{X}| \\ &= |(\mathbb{P}\mathbb{M}\mathbb{P}^{-1})^n\mathbb{P}\hat{X}| = |\hat{\mathbb{M}}^n\mathbb{P}\hat{X}| \\ &= {\lambda_1}^n\{\sum_{j=1}^{M} P_{1j}\hat{X}_j + \sum_{j=1}^{M} q_{n,j}\hat{X}_j\} \end{aligned}$$

where

$$\lim_{n\to\infty} q_{n,j} = 0, \tag{2.2.88}$$

for all $1 \le j \le M$, by (2.2.84).

Hence, by (2.2.87) and (2.2.88),

$$\lim_{n\to\infty} \frac{1}{n} \log \rho(|\mathbb{M}^n \hat{X}|) = \log \lambda_1. \tag{2.2.89}$$

Combining with (2.2.82), (2.2.83) and (2.2.89), (2.2.80) follows.

(ii) $\mathbb{M}$ is irreducible.

If $\mathbb{M}$ is irreducible but imprimitive, then $k \geq 2$ exists, such that

$$\lambda_1 = |\lambda_2| = \cdots = |\lambda_k| > |\lambda_j|$$

for all $j > k$. Then, by applying a permutation, $\mathbb{M}$ can be expressed as

$$\mathbb{M} = \begin{bmatrix} 0 & M_{12} & 0 & \cdots & 0 \\ 0 & 0 & M_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & \vdots & 0 & M_{k-1,k} \\ M_{k1} & 0 & \cdots & \cdots & 0 \end{bmatrix}, \tag{2.2.90}$$

and,

$$\mathbb{M}^k = \begin{bmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & M_k \end{bmatrix}, \tag{2.2.91}$$

where $M_j = M_{j,j+1} M_{j+1,j+2} \cdots M_{j-1,j}$ is primitive with the maximum eigenvalue λ_1^k, see [26], [27]. Hence, by the same argument as in (i)

$$\lim_{n\to\infty} \frac{1}{n} \log \rho(|\mathbb{M}^{nk} \hat{X}|) = \lambda_1^k,$$

(2.2.80) follows.

(iii) $\mathbb{M}$ is reducible.

In this case, by applying a permutation, $\mathbb{M}$ can be expressed as a block upper triangular matrix:

$$\mathbb{M} = \begin{bmatrix} M_{11} & M_{12} & \cdots & \cdots & M_{1k} \\ 0 & M_{22} & \cdots & \cdots & M_{2k} \\ 0 & 0 & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & 0 & M_{kk} \end{bmatrix}, \tag{2.2.92}$$

where M_{ii} is either irreducible or zero. Furthermore,

$$\sigma(\mathbb{M}) = \bigcup_{j=1}^{k} \sigma(\mathbb{M}_{jj}),$$

where $\sigma(\mathbb{M})$ and $\sigma(\mathbb{M}_{jj})$ are the sets of eigenvalues of $\mathbb{M}$ and $\mathbb{M}_{jj}$, respectively. In particular, $1 \leq j \leq k$ exists, such that

$$\rho(\mathbb{M}_{jj}) = \rho(\mathbb{M}) = \lambda_1. \tag{2.2.93}$$

[26], [27]. Therefore, applying (2.2.83), (2.2.93) and the same argument as in (ii) yields (2.2.80).

The proof is complete.

□

Definition 2.13. Let $\mathcal{D}$ denote the set of all diagonal cycle:

$$\mathcal{D} = \{\beta_1\beta_2\cdots\beta_K\beta_{K+1} | \beta_1\beta_2\cdots\beta_K\beta_{K+1} \text{ satisfies (2.2.67) and (2.2.68)}\},$$

define

$$h_*(\mathbb{A}_2) = \sup_{m\geq 2, \beta_1\beta_2\cdots\beta_{K+1}\in\mathcal{D}} \frac{1}{mK} \log \rho(S_{m;\beta_1\beta_2} S_{m;\beta_2\beta_3} \cdots S_{m;\beta_K\beta_1}). \tag{2.2.94}$$

and

$$h'_*(\mathbb{A}_2) = \sup_{m\geq 2,\ \beta_1\cdots\beta_K\in D} \frac{1}{mK} \log \rho(W_{m;\beta_1\beta_2} W_{m;\beta_2\beta_3} \cdots W_{m;\beta_K\beta_1}). \tag{2.2.95}$$

Then Theorem 2.12 implies

$$h(\mathbb{A}_2) \geq h_*(\mathbb{A}_2) \text{ and } h(\mathbb{A}_2) \geq h'_*(\mathbb{A}_2). \tag{2.2.96}$$

Knowing whether the equality holds for $\mathbb{A}_2$ is of interest, since $h_*(\mathbb{A}_2)$ and $h'_*(\mathbb{A}_2)$ are more manageable than $h(\mathbb{A}_2)$. However, a class of $\mathbb{A}_2$ has been found for what equality (2.2.96) holds; details can be found in Example 2.14. of the next subsection.

2.2.3 Examples of transition matrices with positive entropy

In this subsection, various examples are studied to elucidate the power of Theorem 2.12 in verifying that the entropies are positive. First, Golden-Mean type transition matrices are studied.

Example 2.14. (A) Golden-Mean

When two symbols on two-cell horizontal lattice $\mathbb{Z}_{2\times 1}$ and vertical lattice $\mathbb{Z}_{1\times 2}$ are considered and both transition matrices are given by golden-mean, i.e.,

$$\mathbb{H}_1 = \mathbb{V}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

then the (horizontal) transition matrix $\mathbb{A}_2$ on $\mathbb{Z}_{2\times 2}$ is

$$\mathbb{A}_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.2.97}$$

as in [41]. Verifying

$$\mathbb{B}_2 = \widetilde{\mathbb{A}}_2 = \widetilde{\mathbb{B}}_2 = \mathbb{A}_2. \tag{2.2.98}$$

is also easy. Furthermore, for any $n \geq 2$,

$$\mathbb{A}_{n+1} = \begin{bmatrix} A_{n+1} & B_{n+1} \\ C_{n+1} & 0 \end{bmatrix} = \begin{bmatrix} A_n & B_n & A_n & 0 \\ C_n & 0 & C_n & 0 \\ A_n & B_n & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.2.99}$$

where

$$A_{n+1} = \begin{bmatrix} A_n & B_n \\ C_n & 0 \end{bmatrix}$$

with $C_n = {B_n}^t$ and ${A_n}^t = A_n$, i.e., $\mathbb{A}_n$ are symmetric for all $n \geq 2$.

Moreover, the following two properties hold:

(i) For any $m \geq 2$,

$$C_{m;11} = \mathbb{A}_{m-1}, \tag{2.2.100}$$

where

$$\mathbb{A}_1 \equiv \begin{bmatrix} a_{11}a_{11} & a_{12}a_{21} \\ a_{13}a_{31} & a_{14}a_{41} \end{bmatrix}, \tag{2.2.101}$$

and

(ii) for any $m \geq 2$,

$$\frac{1}{m}\log \rho(\mathbb{A}_{m-1}) \leq h(\mathbb{A}_2) \leq \frac{1}{m}\log \rho(\mathbb{A}_m). \tag{2.2.102}$$

Therefore,

$$h(\mathbb{A}_2) = h_*(\mathbb{A}_2) > 0. \tag{2.2.103}$$

The numerical results appears in Example 2.29.

(B) Simplified Golden-Mean.

Consider

$$\mathbb{A}_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.2.104}$$

(2.2.104) cannot be generated from one-dimensional transition matrices $\mathbb{H}_1$ and $\mathbb{V}_1$, as in the Golden-Mean (2.2.97). Equation (2.2.104) is obtained by letting $a_{23} = a_{32} = 0$ in the Golden-Mean (2.2.97). (2.2.98) is easily verified, and for any $n \geq 2$,

$$\mathbb{A}_{n+1} = \begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{$\mathbb{A}_n$}} & \mathbb{A}_{n-1} & 0 \\ & & 0 & 0 \\ \mathbb{A}_{n-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.2.105}$$

Furthermore, (i), (ii) and (2.2.103) hold as in (A).

(C) Generally, if $\mathbb{A}_2$ satisfies the following three conditions

(C1) $\mathbb{B}_2 = \mathbb{A}_2$,

(C2) $a_{1j} = 1$ if $A_{2;j} \neq 0$ for $1 \leq j \leq 4$,

(C3) $\widetilde{A}_{2;1} \geq A_{2;j}$ for $1 \leq j \leq 4$,

then (i), (ii) and (2.2.103) hold. The matrices $\mathbb{A}_2$, which satisfy (C1), (C2) and (C3) can be listed as

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & a_{23} & 0 \\ 1 & a_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.2.106}$$

and

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & a_{23} & a_{24} \\ 1 & a_{32} & 1 & a_{34} \\ 1 & a_{34} & a_{43} & a_{44} \end{bmatrix}, \tag{2.2.107}$$

where a_{ij} is either 0 or 1 in (2.2.106) and (2.2.107).

Notably, if (C2) and (C3) are replaced by

(C2)′ $a_{4j} = 1$ if $A_{2;j} \neq 0$ for $1 \leq j \leq 4$,

(C3)′ $\widetilde{A}_{2;4} \geq A_{2;j}$ for $1 \leq j \leq 4$,

then for any $m \geq 2$,

$$C_{m;44} = \mathbb{A}_{m-1} \tag{2.2.108}$$

with

$$\mathbb{A}_1 = \begin{bmatrix} a_{41}a_{14} & a_{42}a_{24} \\ a_{43}a_{34} & a_{44}a_{44} \end{bmatrix}, \tag{2.2.109}$$

and property (ii) and equation (2.2.103) hold.

In Example 2.14, the diagonal parts $A_{2;1}$ or $A_{2;4}$ are dominant. In this case, only $C_{m;11}$ or $C_{m;44}$ is required to apply Theorem 2.12. In contrast, when $\mathbb{A}_{2;1}$ and $\mathbb{A}_{2;4}$ are no longer dominant as in the following examples, $A_{2;2}$ and $A_{2;3}$ can complement each other to establish that the entropy is positive.

Example 2.15. (A) Consider

$$\mathbb{A}_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.2.110}$$

that (2.2.98) holds can be verified and

$$C_{2;11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C_{2;22} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$
$$C_{2;33} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad C_{2;44} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Therefore,

$$S_{2;14}S_{2;41} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and

$$h(\mathbb{A}_2) \geq \frac{1}{4}\log 2.$$

(B) Consider

$$\mathbb{A}_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}. \tag{2.2.111}$$

Then verifying

$$\mathbb{B}_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad \widetilde{\mathbb{B}}_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad \text{and } \widetilde{\mathbb{A}}_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

is simple.

Furthermore,

$$C_{2;11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C_{2;22} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$C_{2;33} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_{2;44} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$U_{2;11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad U_{2;22} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$U_{2;33} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad U_{2;44} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Now, for any diagonal cycle, $\beta_1 \cdots \beta_K \beta_1$, $\rho(S_{2;\beta_1\beta_2} \cdots S_{2;\beta_K\beta_1}) = 1$, $h(\mathbb{A}_2) > 0$ cannot be established.

However,

$$W_{2;11}W_{2;14}W_{2;41} = U_{2;11}U_{2;22}U_{2;33} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

which implies

$$h(\mathbb{A}_2) \geq \frac{1}{6}\log g,$$

where

$$g = \frac{1}{2}(1+\sqrt{5}) \tag{2.2.112}$$

is the golden mean, which is a root of $\lambda^2 - \lambda - 1 = 0$.

This example demonstrates the asymmetry of $\mathbb{A}_2$ and $\mathbb{B}_2$ in applying Theorem 2.12, to verify the entropy is positive. Both $\mathbb{C}_m$ and $\mathbb{U}_m$ are typically checked for completeness.

Example 2.16. Consider

$$\mathbb{A}_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{2.2.113}$$

Then it is easy to check that

$$W_{2;11}W_{2;14}W_{2;41} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad S_{3;44} = \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$S_{4;44} = \begin{bmatrix} G & 0 & 0 & 0 \\ 0 & e_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$G = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.2.114}$$

Therefore,

$$h(\mathbb{A}_2) \geq \max\{\frac{1}{6}\log 2, \frac{1}{3}\log g, \frac{1}{4}\log g\} = \frac{1}{3}\log g.$$

Example 2.17. Consider

$$\mathbb{A}_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{2.2.115}$$

Then

$$\mathbb{B}_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \tilde{\mathbb{A}}_2 \text{ and } \tilde{\mathbb{B}}_2 = \mathbb{A}_2.$$

Therefore

$$C_{2,11} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \equiv G'.$$

Furthermore,

$$C_{4;11} = G' \otimes e_1 \otimes G'$$

and

$$C_{2m;11} = G' \otimes (\otimes (e_1 \otimes G')^{m-1})$$

can be proved, and which implies

$$\frac{1}{2m} \log \rho(C_{2m;11}) = \frac{1}{2} \log g. \tag{2.2.116}$$

for all $m \geq 1$. Hence, $h(\mathbb{A}_2) \geq \frac{1}{2} \log g$. Moreover, in Remark 2.27 (ii), it can be shown that $h(\mathbb{A}_2) = \frac{1}{2} \log g$

§ 2.3 Trace operators

2.3.1 Trace operator $\mathbb{T}_m$

The preceding section introduces connecting operators $\mathbb{C}_m$, which can be used to find lower bounds of spatial entropy. This section studies the diagonal part of $\mathbb{C}_m$, which can be used to investigate the trace of $\mathbb{A}_n^m$. When $\mathbb{A}_2$ is symmetric, $\mathbb{T}_{2m}$ gives the upper bound of spatial entropy.

The trace operator is defined first.

Definition 2.18. For $m \geq 2$, the m-th order trace operator $\mathbb{T}_m$ of $\mathbb{A}_2$ is defined by

$$\mathbb{T}_m = \begin{bmatrix} C_{m;11} & C_{m;22} \\ C_{m;33} & C_{m;44} \end{bmatrix} = \begin{bmatrix} S_{m;11} & S_{m;14} \\ S_{m;41} & S_{m;44} \end{bmatrix}, \tag{2.3.1}$$

where $C_{m;ij}$ is as given in (2.1.23) or (3.2.45).

Similarly, the m-th order trace operator $\mathbb{T}'_m$ of $\mathbb{B}_2$ is defined by

$$\mathbb{T}'_m = \begin{bmatrix} U_{m;11} & U_{m;22} \\ U_{m;33} & U_{m;44} \end{bmatrix} = \begin{bmatrix} W_{m;11} & W_{m;14} \\ W_{m;41} & W_{m;44} \end{bmatrix} \tag{2.3.2}$$

where $U_{m;ij}$ is as given in (3.2.47).

The relationships between the trace operator $\mathbb{T}_m$, $\mathbb{T}'_m$ and $\mathbb{A}_m$, $\mathbb{B}_m$ are given as follows.

Theorem 2.19. For any $m \geq 2$,
(2.3.3)

$$\mathbb{T}_m = (\mathbb{B}_m)_{2^m\times 2^m} \circ \begin{bmatrix} E_{2^{m-2}\times 2^{m-2}} \otimes \begin{bmatrix} a_{11} & a_{21} \\ a_{31} & a_{41} \end{bmatrix} & E_{2^{m-2}\times 2^{m-2}} \otimes \begin{bmatrix} a_{12} & a_{22} \\ a_{32} & a_{42} \end{bmatrix} \\ \\ E_{2^{m-2}\times 2^{m-2}} \otimes \begin{bmatrix} a_{13} & a_{23} \\ a_{33} & a_{43} \end{bmatrix} & E_{2^{m-2}\times 2^{m-2}} \otimes \begin{bmatrix} a_{14} & a_{24} \\ a_{34} & a_{44} \end{bmatrix} \end{bmatrix}$$

and
(2.3.4)

$$\mathbb{T}'_m = (\mathbb{A}_m)_{2^m\times 2^m} \circ \begin{bmatrix} E_{2^{m-2}\times 2^{m-2}} \otimes \begin{bmatrix} b_{11} & b_{21} \\ b_{31} & b_{41} \end{bmatrix} & E_{2^{m-2}\times 2^{m-2}} \otimes \begin{bmatrix} b_{12} & b_{22} \\ b_{32} & b_{42} \end{bmatrix} \\ \\ E_{2^{m-2}\times 2^{m-2}} \otimes \begin{bmatrix} b_{13} & b_{23} \\ b_{33} & b_{43} \end{bmatrix} & E_{2^{m-2}\times 2^{m-2}} \otimes \begin{bmatrix} b_{14} & b_{24} \\ b_{34} & b_{44} \end{bmatrix} \end{bmatrix}.$$

In particular,

$$\mathbb{T}_m \leq \mathbb{B}_m \text{ and } \mathbb{T}'_m \leq \mathbb{A}_m. \tag{2.3.5}$$

Proof. By (3.3.1) and (3.2.45),

$$\mathbb{T}_m = (\mathbb{B}_m)_{2^m\times 2^m} \circ \begin{bmatrix} E_{2^{m-2}\times 2^{m-2}} \otimes \begin{bmatrix} a_{11} & a_{21} \\ a_{31} & a_{41} \end{bmatrix} & E_{2^{m-2}\times 2^{m-2}} \otimes \begin{bmatrix} a_{12} & a_{22} \\ a_{32} & a_{42} \end{bmatrix} \\ \\ E_{2^{m-2}\times 2^{m-2}} \otimes \begin{bmatrix} a_{13} & a_{23} \\ a_{33} & a_{43} \end{bmatrix} & E_{2^{m-2}\times 2^{m-2}} \otimes \begin{bmatrix} a_{14} & a_{24} \\ a_{34} & a_{44} \end{bmatrix} \end{bmatrix}.$$

A similar result also holds for $\mathbb{T}'_m$. Hence, (3.3.25) follows immediately.

The proof is complete. □

Notably, the trace operator $\mathbb{T}_m$ (or $\mathbb{T}'_m$) preserves all periodic words $a_{i_1i_2}a_{i_2i_3}\cdots a_{i_mi_{m+1}}$ ($b_{i_1i_2}b_{i_2i_3}\cdots b_{i_mi_{m+1}}$) with $i_{m+1} = i_1$ of length m systematically as $\mathbb{B}_m$ (or $\mathbb{A}_m$).

The traces of the elementary patterns are defined accordingly.

Definition 2.20. For $m, n \geq 2$ and $1 \leq \alpha \leq 4$, define

$$t^{(k)}_{m,n;\alpha} = tr(A^{(k)}_{m,n;\alpha}), \tag{2.3.6}$$

$$tr(X_{m,n;\alpha}) = (t^{(k)}_{m,n;\alpha})_{1\leq k\leq 2^{m-1}}, \tag{2.3.7}$$

and

$$t_{m,n} = (tr(X_{m,n;1}), tr(X_{m,n;4}))^t, \tag{2.3.8}$$

which are 2^{m-1} and 2^m vectors, respectively.

Note that

$$\begin{array}{rcl} tr(\mathbb{A}_n^m) & = & tr(\sum_{k=1}^{2^{m-1}} A_{m,n;1}^{(k)} + \sum_{k=1}^{2^{m-1}} A_{m,n;4}^{(k)}) \\ & = & |tr(X_{m,n;1})| + |tr(X_{m,n;4})| \\ & = & |t_{m,n}|. \end{array} \tag{2.3.9}$$

First prove that $\mathbb{T}_m$ can reduce the traces of higher-order to lower-order.

Proposition 2.21. For $m \geq 2$ and $n \geq 2$,

$$t_{m,n+1} = \mathbb{T}_m t_{m,n} \tag{2.3.10}$$

Proof. By Theorem 2.5, it is easy to see

$$\begin{pmatrix} tr(X_{m,n+1;1}) \\ \\ tr(X_{m,n+1;4}) \end{pmatrix} = \begin{pmatrix} C_{m;11} tr(X_{m,n;1}) + C_{m;22} tr(X_{m,n;4}) \\ \\ C_{m;33} tr(X_{m,n;1}) + C_{m;44} tr(X_{m,n;4}) \end{pmatrix}.$$

Then, (3.3.16) follows immediately.

The proof is complete. □

Repeatedly applying Proposition 2.21 yields the following result.

Theorem 2.22. For $m \geq 2$ and $n \geq 1$,

$$tr(\mathbb{A}_{n+2}^m) = |\mathbb{T}_m^n t_{m,2}| \tag{2.3.11}$$

$$\equiv \sum_{\beta_k \in \{1,4\}} |S_{m;\beta_1\beta_2} S_{m;\beta_2\beta_3} \cdots S_{m;\beta_n\beta_{n+1}} tr(X_{m,2;\beta_{n+1}})|. \tag{2.3.12}$$

Proof.

$$\begin{aligned} & tr(\mathbb{A}_n^m) \\ & = \sum_{k=1}^{2^{m-1}} tr(A_{m,n;1;1}^{(k)}) + \sum_{k=1}^{2^{m-1}} tr(A_{m,n;1;4}^{(k)}) + \sum_{k=1}^{2^{m-1}} tr(A_{m,n;4;1}^{(k)}) + \sum_{k=1}^{2^{m-1}} tr(A_{m,n;4;4}^{(k)}) \\ & = |tr(X_{m,n;1;1})| + |tr(X_{m,n;1;4})| + |tr(X_{m,n;4;1})| + |tr(X_{m,n;4;4})| \\ & = |tr(S_{m;11} X_{m,n-1;1})| + |tr(S_{m;14} X_{m,n-1;4})| + |tr(S_{m;41} X_{m,n-1;1})| + |tr(S_{m;44} X_{m,n-1;4})| \\ & = |\mathbb{T}_m t_{m,n-1}|, \end{aligned}$$

here Theorem 2.4 is used.

Reduction on n, yields

$$tr(\mathbb{A}_n^m) = |\mathbb{T}_m^{n-2} t_{m,2}|.$$

Finally, (3.3.18) follows from (3.3.1) and (3.3.11).

The proof is complete. □

The following lemma is needed to show (2.1.33).

Lemma 2.23. Let V_m be a nonnegative eigenvector of $\mathbb{T}_m$ with respect to the maximum eigenvalue $\rho(\mathbb{T}_m)$. If $\rho(\mathbb{T}_m) > 0$, then

$$\langle V_m, t_{m,2}\rangle > 0,$$

where $\langle\ ,\ \rangle$ denotes the standard inner product of $\mathbb{C}^{2^m}$.

Proof. Let $V_m = (u_1, \cdots, u_M, u'_1, \cdots, u'_M)$ be a nonnegative eigenvector of $\mathbb{T}_m$, where $M = 2^{m-1}$. Since $\rho(\mathbb{T}_m) > 0$, by Lemma 2.11, if $u_k > 0$ (or $u'_l > 0$) then $tr(A^{(k)}_{m,2;1}) > 0$ (or $tr(A^{(l)}_{m,2;4}) > 0$). The result follows by (3.3.11).

The proof is complete. □

Now, (2.1.33) can be proved.

Theorem 2.24. For any $m \geq 2$,

$$\limsup_{n\to\infty} \frac{1}{n} \log tr(\mathbb{A}_n^m) = \log \rho(\mathbb{T}_m), \tag{2.3.13}$$

and

$$h(\mathbb{A}_2) = \limsup_{m\to\infty} \frac{1}{m} \log \rho(\mathbb{T}_m). \tag{2.3.14}$$

Furthermore, if $\mathbb{A}_n$ are primitive for all $n \geq 2$, then *limsup* in (3.3.19) and (3.3.20) can be replaced by *lim*, i.e.,

$$\lim_{n\to\infty} \frac{1}{n} \log tr(\mathbb{A}_n^m) = \log \rho(\mathbb{T}_m) \tag{2.3.15}$$

and

$$h(\mathbb{A}_2) = \lim_{m\to\infty} \frac{1}{m} \log \rho(\mathbb{T}_m). \tag{2.3.16}$$

Proof. By Perron-Frobenius theorem, for all $n \geq 2$, we have

$$\limsup_{m\to\infty} \frac{1}{m} \log tr(\mathbb{A}_n^m) = \log \rho(\mathbb{A}_n). \tag{2.3.17}$$

Therefore, by (2.3.17) and Theorem 2.22, we have

$$h(\mathbb{A}_2) = \lim_{n\to\infty} \frac{1}{n} \log \rho(\mathbb{A}_n) = \limsup_{n,m\to\infty} \frac{1}{mn} \log tr(\mathbb{A}_n^m) = \limsup_{n,m\to\infty} \frac{1}{mn} \log |\mathbb{T}_m^n t_{m,2}|.$$

By Lemma 2.23 and by argument used to prove Theorem 2.12,

$$\limsup_{n\to\infty} \frac{1}{n} \log |\mathbb{T}_m^n t_{m,2}| = \log \rho(\mathbb{T}_m) \tag{2.3.18}$$

can be shown, and (3.3.19) and (3.3.20) follow immediately.

When $\mathbb{A}_n$ are primitive for all $n \geq 2$, (2.3.15) and (2.3.16) follow.

The proof is complete. □

Now, the symmetry of $\mathbb{A}_2$ is established to be able to be inherited by the higher order matrices.

Proposition 2.25. If $\mathbb{A}_2$ is symmetric, then $\mathbb{A}_n$ is also symmetric for each $n \geq 3$.

Proof. The proposition is proven by induction on n.

Let $\mathbb{M} = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}$ be a square matrix and M_i, $1 \leq i \leq 4$, all be square matrices. Then, the transpose matrix $\mathbb{M}^t$ of $\mathbb{M}$ is

$$\mathbb{M}^t = \begin{bmatrix} {M_1}^t & {M_3}^t \\ {M_2}^t & {M_4}^t \end{bmatrix}.$$

Therefore, $\mathbb{M}$ is symmetric if and only if

$${M_1}^t = M_1,\ {M_3}^t = M_2 \text{ and } {M_4}^t = M_4.$$

In particular, $\mathbb{A}_2$ is symmetric if and only if

$$A_{2;1}^t = A_{2;1},\ A_{2;3}^t = A_{2;2} \text{ and } A_{2;4}^t = A_{2;4}. \tag{2.3.19}$$

Now, $\mathbb{A}_n$ is assumed to be symmetric, such that

$$A_{n;1}^t = A_{n;1},\ A_{n;3}^t = A_{n;2} \text{ and } A_{n;4}^t = A_{n;4}. \tag{2.3.20}$$

Since

$$A_{n+1;\alpha} = [A_{2;\alpha}]_{2\times 2} \circ \begin{bmatrix} A_{n;1} & A_{n;2} \\ A_{n;3} & A_{n;4} \end{bmatrix},$$

(3.3.21) and (3.3.21) imply

$$A_{n+1;1}^t = A_{n+1;1},\ A_{n+1;3}^t = A_{n+1;2} \text{ and } A_{n+1;4}^t = A_{n+1;4}.$$

Hence, $\mathbb{A}_{n+1}$ is symmetric.

The proof is complete. □

Now, upper estimates of spatial entropy $h(\mathbb{A}_2)$ are obtained when $\mathbb{A}_2$ is symmetric.

Theorem 2.26. If $\mathbb{A}_2$ is symmetric then for any $m \geq 1$,

$$h(\mathbb{A}_2) \leq \frac{1}{2m} \log \rho(\mathbb{T}_{2m}). \tag{2.3.21}$$

Proof. By Proposition 2.25, $\mathbb{A}_n^{2m}$ is symmetric for any $m \geq 1$. The symmetry of $\mathbb{A}_n^{2m}$ implies that all eigenvalues of $\mathbb{A}_n^{2m}$ are non-negative. Hence,

$$\rho(\mathbb{A}_n)^{2m} = \rho(\mathbb{A}_n^{2m}) \leq tr(\mathbb{A}_n^{2m}). \tag{2.3.22}$$

On the other hand, the subadditivity of (2.2.58) implies

$$h(\mathbb{A}_2) \leq \frac{1}{(2mk+1)n} \log |\mathbb{A}_n^{2mk}|. \tag{2.3.23}$$

Therefore, (3.3.24), (3.3.22) and (3.3.17) imply

$$\begin{aligned} h(\mathbb{A}_2) &\leq \lim_{n,k\to\infty} \frac{1}{(2mk+1)n} \log |\mathbb{A}_n^{2mk}| = \lim_{n\to\infty} \frac{1}{2mn} \log \rho(\mathbb{A}_n^{2m}) \\ &\leq \lim_{n\to\infty} \frac{1}{2mn} \log tr(\mathbb{A}_n^{2m}) = \lim_{n\to\infty} \frac{1}{2mn} \log |\mathbb{T}_{2m}^{n-2} t_{2m,2}| \\ &\leq \frac{1}{2m} \log \rho(\mathbb{T}_{2m}). \end{aligned}$$

The proof is complete. □

Notably, $\mathbb{T}_m$ (or $\mathbb{T}'_m$) yields a better estimate than $\mathbb{B}_n$ (or $\mathbb{A}_n$) whenever

$$h(\mathbb{A}_2) \leq \frac{1}{m} \log \rho(\mathbb{T}_m) \tag{2.3.24}$$

holds.

Remark 2.27. (i) The problem in which $\mathbb{A}_n$ are primitive for all $n \geq 2$ has already been investigated [6]. In [6], various sufficient conditions have been found to ensure that $\mathbb{A}_n$ are primitive for all $n \geq 2$. Notably, limit in (2.3.15) and (2.3.16), instead of *limsup* in (3.3.19) and (3.3.20), causes $\mathbb{A}_n$ to have a unique maximum eigenvalue with a maximum modulus. Therefore, $\mathbb{A}_n$ may be imprimitive but (2.3.15) and (2.3.16) still hold. For example, Golden-Mean and simplified Golden-Mean in Example 2.14 are imprimitive but (2.3.15) and (2.3.16) still hold. The remaining matrices of these $\mathbb{A}_n$ are primitive if their rows and columns with zero entries are removed.

(ii) In general, *limsup* cannot be replaced by *limit*. For example, consider

$$\mathbb{A}_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{2.3.25}$$

Further computation shows that

$$\mathbb{T}_{2m+1} = 0$$

and

$$\mathbb{T}_{2m} = \begin{bmatrix} (\otimes(G^{'} \otimes e_1)^{m-1}) \otimes G^{'} & e_1 \otimes (\otimes(G^{'} \otimes e_1)^{m-1}) \\ e_1 \otimes (\otimes(G^{'} \otimes e_1)^{m-1}) & e_1 \otimes (\otimes(G^{'} \otimes e_1)^{m-1}) \end{bmatrix}$$

for all $m \geq 1$, where $G^{'} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Therefore, $\rho(\mathbb{T}_{2m+1}) = 0$. Furthermore, it can be shown that

$$\rho(\mathbb{T}_{2m}) \leq g^m + g^{m-1}. \tag{2.3.26}$$

Combining (2.2.116) and (2.3.26), $h(\mathbb{A}_2) = \frac{1}{2}\log g$. Hence (3.3.20) holds only for *limsup*. Unlike (2.2.62) this example demonstrates that (3.3.23) does not hold for any $n = 2m + 1$. This phenomenon is a disadvantage in determining the upper estimate of entropy associated with replacing $\mathbb{A}_n$ with $\mathbb{T}_n$.

Example 2.28. Consider

$$\mathbb{A}_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

which was studied as in Example 3.2.9. Now, $\mathbb{A}_2$ is asymmetric. Furthermore,

$$tr(\mathbb{A}_n^2) = 3$$

can be obtained for all $n \geq 2$. Hence, (3.3.24) and then (3.3.26) fail when $m = 1$. However,

$$C_{4;44} = \begin{bmatrix} G & 0 & 0 & 0 \\ 0 & e_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $G = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence $tr(\mathbb{A}_n^4)$ grows at least exponentially with exponent $\rho(G) = g$, the golden-mean.

Whether (3.3.26) holds for some $m \geq 2$ is of interest.

Example 2.29. Consider the Golden-Mean

$$\mathbb{A}_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which was studied as in Example 2.14. $\mathbb{A}_2$ is symmetric, so the numerical results can be obtained as follows.

m	$\rho(\mathbb{A}_{m-1})^{\frac{1}{m}}$	$\rho(\mathbb{T}_m)^{\frac{1}{m}}$	$\rho(\mathbb{A}_m)^{\frac{1}{m}}$
2	1.3415037626	1.5537739740	1.5537739740
3	1.3804413572	1.4892228485	1.5370592754
4	1.4041128626	1.5069022259	1.5284545258
5	1.4201397131	1.5017251916	1.5233415461
6	1.4316975290	1.5035148094	1.5199401525
7	1.4404277508	1.5028716910	1.5175154443
8	1.4472546963	1.5031163748	1.5156994341
9	1.4527395436	1.5030208210	1.5142884861
10	1.4572426033	1.5030591603	1.5131606734
11	1.4610058138	1.5030435026	1.5122385423
12	1.4641976583	1.5030500001	1.5114705290
13	1.4669390746	1.5030472703	1.5108209763
14	1.4693191202	1.5030484295	1.5102644390
15	1.4714048275	1.5030479329	1.5097822725
16	1.4732476160	1.5030481473	1.5093605030

Notably, both $\rho(\mathbb{A}_m)^{\frac{1}{m}}$ and $\rho(\mathbb{T}_{2m})^{\frac{1}{2m}}$ are monotonically decreasing in m. In contrast, $\rho(\mathbb{A}_{m-1})^{\frac{1}{m}}$ and $\rho(\mathbb{T}_{2m+1})^{\frac{1}{2m+1}}$ are monotonically increasing in m, that $\rho(\mathbb{T}_{2m})^{\frac{1}{2m}}$ gives better upper bound than $\rho(\mathbb{A}_m)^{\frac{1}{m}}$. That $\rho(\mathbb{T}_{2m+1})^{\frac{1}{2m+1}}$ are lower bounds is conjectured. If they were, then $\rho(\mathbb{T}_m)^{\frac{1}{m}}$ would yield a very sharp estimates.

§ 2.4 More symbols on larger lattice

As mentioned in the introduction, many physical and engineering problems involve many (more than two) symbols and larger lattices. Therefore, the results found in the previous sections must be extended to any finite number of symbols $p \geq 2$ on any finite square lattice $\mathbb{Z}_{2l\times 2l,\ l\geq 1}$. The results are only outlined here, and the details are left to the readers. Proofs of theorems are omitted for brevity.

For fixed $p \geq 2$ and $l \geq 1$, denote by

$$q = p^{l^2}. \tag{2.4.1}$$

The horizontal and vertical transition matrices are given by

$$\mathbb{A}_2 = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,q^2} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,q^2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{q^2,1} & a_{q^2,2} & \cdots & a_{q^2,q^2} \end{bmatrix} \tag{2.4.2}$$

and

$$\mathbb{B}_2 = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,q^2} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,q^2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{q^2,1} & b_{q^2,2} & \cdots & b_{q^2,q^2} \end{bmatrix}, \tag{2.4.3}$$

respectively.
Now, $\mathbb{A}_2$ and $\mathbb{B}_2$ are related to each other by

$$\mathbb{A}_2 = \begin{bmatrix} A_{2;1} & A_{2;2} & \cdots & A_{2;q} \\ A_{2;q+1} & A_{2;q+2} & \cdots & A_{2;2q} \\ \vdots & \vdots & \ddots & \vdots \\ A_{2;q(q-1)+1} & \cdots & \cdots & A_{2;q^2} \end{bmatrix} \tag{2.4.4}$$

where

$$A_{2;\alpha} = \begin{bmatrix} b_{\alpha,1} & b_{\alpha,2} & \cdots & b_{\alpha,q} \\ b_{\alpha,q+1} & b_{\alpha,q+2} & \cdots & b_{\alpha,2q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{\alpha,q(q-1)+1} & b_{\alpha,q(q-1)+2} & \cdots & b_{\alpha,q^2} \end{bmatrix}, \tag{2.4.5}$$

and

$$\mathbb{B}_2 = \begin{bmatrix} B_{2;1} & B_{2;2} & \cdots & B_{2;q} \\ B_{2;q+1} & B_{2;q+2} & \cdots & B_{2;2q} \\ \vdots & \vdots & \ddots & \vdots \\ B_{2;q(q-1)+1} & \cdots & \cdots & B_{2;q^2} \end{bmatrix} \tag{2.4.6}$$

where

$$B_{2;\alpha} = \begin{bmatrix} a_{\alpha,1} & a_{\alpha,2} & \cdots & a_{\alpha,q} \\ a_{\alpha,q+1} & a_{\alpha,q+2} & \cdots & a_{\alpha,2q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\alpha,q(q-1)+1} & a_{\alpha,q(q-1)+2} & \cdots & a_{\alpha,q^2} \end{bmatrix}, \tag{2.4.7}$$

respectively, where $1 \leq \alpha \leq q^2$. The column matrices $\widetilde{\mathbb{A}_2}$ and $\widetilde{\mathbb{B}_2}$, $\mathbb{A}_2$ and $\mathbb{B}_2$ are defined as in (3.2.2) and (3.2.3). For higher order transition matrices $\mathbb{A}_n$, $n \geq 3$, are defined as

$$\mathbb{A}_n = \begin{bmatrix} A_{n;1} & A_{n;2} & \cdots & A_{n;q} \\ A_{n;q+1} & A_{n;q+2} & \cdots & A_{n;2q} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n;q(q-1)+1} & A_{n;(q-1)q+2} & \cdots & A_{n;q^2} \end{bmatrix} \tag{2.4.8}$$

where
(2.4.9)

$$\mathbb{A}_{n;\alpha} = \begin{bmatrix} b_{\alpha,1}A_{n-1;1} & b_{\alpha,2}A_{n-1;2} & \cdots & b_{\alpha,q}A_{n-1;q} \\ b_{\alpha,q+1}A_{n-1;q+1} & b_{\alpha,q+2}A_{n-1;q+2} & \cdots & b_{\alpha,2q}A_{n-1;2q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{\alpha,q(q-1)+1}A_{n-1;q(q-1)+1} & b_{\alpha,q(q-1)+2}A_{n-1;q(q-1)+2} & \cdots & b_{\alpha,q^2}A_{n;q^2} \end{bmatrix}.$$

Rewriting the indices of $A_{n;\alpha}$ as follows, facilitates matrix multiplication.

$$\mathbb{A}_n = \begin{bmatrix} A_{n;11} & A_{n;12} & \cdots & A_{n;1q} \\ A_{n;21} & A_{n;22} & \cdots & A_{n;2q} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n;q1} & A_{n;q2} & \cdots & A_{n;qq} \end{bmatrix}. \tag{2.4.10}$$

Clearly, $A_{n;\alpha} = A_{n;j_1j_2}$, where

$$\alpha = \alpha(j_1, j_2) = q(j_1 - 1) + j_2. \tag{2.4.11}$$

For $m \geq 2$, the elementary pattern in the entries of $\mathbb{A}_n^m$ is given by

$$A_{n;j_1j_2}A_{n;j_2j_3}\cdots A_{n;j_mj_{m+1}},$$

where $j_s \in \{1, 2, \cdots, q\}$.

The lexicographic order for multiple indices

$$J_{m+1} = (j_1j_2\cdots j_mj_{m+1})$$

is introduced by

$$\chi(J_{m+1}) = 1 + \sum_{l=2}^{m} q^{m-l}(j_l - 1). \tag{2.4.12}$$

Specify

$$A^{(k)}_{m,n;\alpha} = A_{n;j_1j_2}A_{n;j_2j_3}\cdots A_{n;j_mj_{m+1}},$$

where $\alpha = \alpha(j_1, j_{m+1})$ satisfies (3.4.6) and $k = \chi(J_{m+1})$ is as given in (3.4.7). Based on this arrangement, $\mathbb{A}^m_n$ can be written as

$$\mathbb{A}^m_n = \begin{bmatrix} A_{m,n;1} & A_{m,n;2} & \cdots & A_{m,n;q} \\ A_{m,n;q+1} & A_{m,n;q+2} & \cdots & A_{m,n;2q} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m,n;q(q-1)+1} & A_{m,n;q(q-1)+2} & \cdots & A_{m,n;q^2} \end{bmatrix},$$

where

$$A_{m,n;\alpha} = \sum_{k=1}^{q^{m-1}} A^{(k)}_{m,n;\alpha}.$$

Moreover, $X_{m,n;\alpha} = (A^{(k)}_{m,n;\alpha})^t$, where $1 \le k \le q^{m-1}$ and $X_{m,n;\alpha}$ is a q^{m-1}-vector that comprise all elementary patterns in $A_{m,n;\alpha}$. The ordering matrix $\mathbb{X}_{m,n}$ of $\mathbb{A}^m_n$ is now defined as

$$\mathbb{X}_{m,n} = \begin{bmatrix} X_{m,n;1} & X_{m,n;2} & \cdots & X_{m,n;q} \\ X_{m,n;q+1} & X_{m,n;q+2} & \cdots & X_{m,n;2q} \\ \vdots & \vdots & \ddots & \vdots \\ X_{m,n;q(q-1)+1} & X_{m,n;q(q-1)+2} & \cdots & X_{m,n;q^2} \end{bmatrix},$$

and $X_{m,n+1;\beta}$ can be reduced to $X_{2,n;\beta}$ by multiplication with connecting matrices $C_{m;\alpha,\beta}$. The connecting operator $\mathbb{C}_m$ is defined as follows.

Definition 2.30. For $m \ge 2$, define

$$\mathbb{C}_m = \begin{bmatrix} C_{m;1,1} & C_{m;1,2} & \cdots & C_{m;1,q^2} \\ C_{m;2,1} & C_{m;2,2} & \cdots & C_{m;2,q^2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m;q^2,1} & C_{m;q^2,2} & \cdots & C_{m;q^2,q^2} \end{bmatrix}$$

(2.4.13)

$$
= \left[\begin{array}{ccc|c|ccc}
S_{m;1,1} & \cdots & S_{m;1,q} & & S_{m;q,1} & \cdots & S_{m;q,q} \\
\vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\
S_{m;1,q(q-1)+1} & \cdots & S_{m;1,q^2} & & S_{m;q,q(q-1)+1} & \cdots & S_{m;q,q^2} \\
\hline
 & \vdots & & \ddots & & \vdots & \\
\hline
S_{m;q(q-1)+1,1} & \cdots & S_{m;q(q-1)+1,q} & & S_{m;q^2,1} & \cdots & S_{m;q^2,q} \\
\vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\
S_{m;q(q-1)+1,q(q-1)+1} & \cdots & S_{m;q(q-1)+1,q^2} & & S_{m;q^2,q(q-1)+1} & \cdots & S_{m;q^2,q^2}
\end{array}\right]
$$

where

(2.4.14)

$$C_{m;\alpha,\beta} = ((B_{2;\alpha})_{q\times q} \circ (\hat{\otimes}\mathbb{B}_2^{m-2})_{q\times q})_{q^{m-1}\times q^{m-1}} \circ (E_{q^{m-2}\times q^{m-2}} \otimes \tilde{A}_{2;\beta})_{q^{m-1}\times q^{m-1}}.$$

Like Theorem 2.4, $C_{m+1;\alpha,\beta}$ can be obtained in terms of $C_{m;\gamma,\beta}$.

Theorem 2.31. For any $m \geq 2$ and $1 \leq \alpha, \beta \leq q^2$

$$
C_{m+1;\alpha,\beta} = \begin{bmatrix}
a_{\alpha;1}C_{m;1,\beta} & a_{\alpha;2}C_{m;2,\beta} & \cdots & a_{\alpha;q}C_{m;q,\beta} \\
a_{\alpha;q+1}C_{m;q+1,\beta} & a_{\alpha;q+2}C_{m;q+2,\beta} & \cdots & a_{\alpha;2q}C_{m;2q,\beta} \\
\vdots & \vdots & \ddots & \vdots \\
a_{\alpha;q(q-1)+1}C_{m;q(q-1)+1,\beta} & a_{\alpha;q(q-1)+2}C_{m;q(q-1)+2,\beta} & \cdots & a_{\alpha;q^2}C_{m;q^2,\beta}
\end{bmatrix}.
$$

Denote by

$$
A^{(k)}_{m,n+1;\alpha} = \begin{bmatrix}
A^{(k)}_{m,n+1;\alpha;1} & A^{(k)}_{m,n+1;\alpha;2} & \cdots & A^{(k)}_{m,n+1;\alpha;q} \\
A^{(k)}_{m,n+1;\alpha;q+1} & A^{(k)}_{m,n+1;\alpha;q+2} & \cdots & A^{(k)}_{m,n+1;\alpha;2q} \\
\vdots & \vdots & \ddots & \vdots \\
A^{(k)}_{m,n+1;\alpha;q(q-1)+1} & A^{(k)}_{m,n+1;\alpha;q(q-1)+2} & \cdots & A^{(k)}_{m,n+1;\alpha;q^2}
\end{bmatrix}
$$

and $X_{m,n+1;\alpha;\beta} = (A^{(k)}_{m,n+1;\alpha;\beta})^t$ where $A^{(k)}_{m,n+1;\alpha;\beta}$ is a linear combination of $A^{(l)}_{m,n;\gamma}$. Now, Theorem 2.5 can be generalized to the following theorem.

Theorem 2.32. For any $m \geq 2$ and $n \geq 2$, let $S_{m;\alpha,\beta}$ be as given in (3.4.8) and (3.4.9). Then $X_{m,n+1;\alpha;\beta} = S_{m;\alpha,\beta}X_{m,n;\beta}$.

Reference

[1] J.C. Ban, K.P. Chien and S.S. Lin, *Spatial disorder of CNN-with asymmetric output funtion*, International J. of Bifurcation and Chaos **11**, (2001) 2085-2095.

[2] J.C. Ban, C.H. Hsu AND S.S.Lin, *Spatial disorder of cellular neural network-with biased term*, International J. of Bifurcation and Chaos **12**, (2002) 525-534.

[3] J.C. Ban, S.S. Lin and C.W. Shih, *Exact number of mosaic patterns in one-dimensional cellular neural networks*, International J. of Bifurcation and Chaos **11**, (2001) 1645-1653.

[4] J.C. Ban and S.S. Lin, *Patterns generation and transition matrices in multi-dimensional lattice models*, Discrete Contin. Dyn. Syst. **13**, (2005), no. 3, 637-658.

[5] J.C. Ban, S.S. Lin and Y.H. Lin, *Patterns generation and spatial entropy in three-dimensional lattice models*, preprint (2005).

[6] J.C. Ban, S.S. Lin and Y.H. Lin, *Primitivity of subshifts of finite type in two-dimensional lattice models*, preprint (2005).

[7] P.W. Bates and A. Chmaj, *A discrete convolution model for phase transitions*, Arch. Rat. Mech. Anal **150**, (1999) 281-305.

[8] P.W. Bates, K. Lu and B. Wang, *Attractors for lattice dynamical systems*, International J. of Bifurcation and Chaos **11**, (2001) 143-153.

[9] J. Bell, *Some threshold results for modes of myelinated nerves*, Math. Biosci. **54**, (1981) 181-190.

[10] J. Bell and C. Cosner, *Threshold behavior and propagation for nonlinear differential-difference systems motivated by modeling myelinated axons*, Quart. Appl. Math. **42**, (1984) 1-14.

[11] R. J. Baxter, *Eight-vertex model in lattice statistics.*, Phys. Rev. Lett. **26**, (1971) 832-833.

[12] J.W. Cahn, *Theory of crystal growth and interface motion in crystalline materials*, Acta Metallurgica **8**, (1960) 554-562.

[13] S.N. Chow and J. Mallet-paret, *Pattern formation and spatial chaos in lattice dynamical systems II*, IEEE Trans. Circuits Systems **42**, (1995) 752-756.

[14] S.N. Chow, J. Mallet-paret and E. S. Van Vleck, *Dynamics of lattice differential equations*, International J. of Bifurcation and Chaos **6**, (1996) 1605-1621.

[15] S.N. Chow, J. Mallet-paret and E.S. Van Vleck, *Pattern formation and spatial chaos in spatially discrete evolution equations*, Random Comput. Dynam. **4**, (1996) 109-178.

[16] L.O. Chua, *CNN: A paradigm for complexity* (World Scientific Series on Nonlinear Science, Series A,31. World Scietific, Singapore, 1998)

[17] L.O. Chua, K.R. Crounse, M. Hasler and P. Thiran, *Pattern formation properties of autonomous cellular neural networks*, IEEE Trans. Circuits Systems **42**, (1995) 757-774.

[18] L.O. Chua and T. Roska, *The CNN paradigm*, IEEE Trans. Circuits Systems **40**, (1993) 147-156.

[19] L.O. Chua and L. Yang, *Cellular neural networks: Theory*, IEEE Trans. Circuits Systems **35**, (1988) 1257-1272.

[20] L.O. Chua and L. Yang, *Cellular neural networks: Applications*, IEEE Trans. Circuits Systems **35**, (1988) 1273-1290.

[21] G.B. Ermentrout, *Stable periodic solutions to discrete and continuum arrays of weakly coupled nonlinear oscillators*, SIAM J. Appl. Math. **52**, (1992) 1665-1687.

[22] G.B. Ermentrout and N. Kopell, *Inhibition-produced patterning in chains of coupled nonlinear oscillators*, SIAM J. Appl. Math. **54**, (1994) 478-507.

[23] G.B. Ermentrout, N. Kopell and T. L. Williams, *On chains of oscillators forced at one end*, SIAM J. Appl. Math. **51**, (1991) 1397-1417.

[24] T. Eveneux and J.P. Laplante, *Propagation failure in arrays of coupled bistable chemical reactors*, J. Phys. Chem. **96**, (1992) 4931-4934.

[25] W.J. Firth, *Optical memory and spatial chaos*, Phys. Rev. Lett. **61**, (1988) 329-332.

[26] F.R. Gantmacher, *The theory of matrices* (2 vols. Chelsea, New York, 1959)

[27] R.A. Horn and C.R. Johnson, *Matrix analysis* (Cambridge University Press, Cambridge, 1990)

[28] C.H. Hsu, J. Juang, S.S. Lin, and W.W. Lin, *Cellular neural networks: local patterns for general template*, International J. of Bifurcation and Chaos **10**, (2000) 1645-1659.

[29] J. Juang and S.S. Lin, *Cellular Neural Networks: Mosaic pattern and spatial chaos*, SIAM J. Appl. Math. **60**, (2000) 891-915.

[30] J. Juang, S.S. Lin, W.W. Lin and S.F. Shieh, *The spatial entropy of two-dimensional subshifts of finite type*, International J. of Bifurcation and Chaos **10**, (2000) 2845-2852.

[31] J.P. Keener, *Propagation and its failure in coupled systems of discrete excitable cells*, SIAM J. Appl. Math. **47**, (1987) 556-572.

[32] J.P. Keener, *The effects of discrete gap junction coupling on propagation in myocardium*, J. Theor. Biol. **148**, (1991) 49-82.

[33] A.L. Kimball, A. Varghese and R.L. Winslow, *Simulating cardiac sinus and atrial netwok dynamics on the connection machine*, Phys. D **64**, (1993) 281-298.

[34] E.H. Lieb, *Exact solution of the problem of the entropy of two-dimensional ice*, Phys. Rev. Lett. **18**, (1967) 692-694.

[35] E.H. Lieb, *Exact solution of the f model of an antiferroelectric*, Phys. Rev. Lett. **18**, (1967) 1046-1048.

[36] E.H. Lieb, *Exact Solution of the Two-Dimensional Slater KDP Model of a Ferroelectric*, Phys. Rev. Lett. **19**, (1967) 108-110.

[37] E.H. Lieb, *The Residual Entropy of Square Ice*, Phys. Rev. **162**, (1967) 162-172.

[38] E.H. Lieb, *Ice, ferro- and antiferroelectrics, in methods and problems in theoretical physics*, in honour of R.E. Peierls, proceedings of the 1967 Birmingham conference, (North-Holland, 1970) 21-28.

[39] S.S. Lin and T.S. Yang, *Spatial entropy of one-dimensional celluar neural network*, International J. of Bifurcation and Chaos **10**, (2000) 2129-2140.

[40] S.S. Lin and T.S. Yang, *On the spatial entropy and patterns of two-dimensional cellular neural networks*, International J. of Bifurcation and Chaos **12**, (2002) 115-128.

[41] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, (1995).

[42] A. Lindenmayer and P. Prusinkiewicz, *The algorithmic beauty of plants*, Springer-Verlag, New York, (1990).

[43] N.G. Markley and M.E. Paul, *Maximal measures and entropy for Z^ν subshifts of finite type.* Classical mechanics and dynamical systems, Medford, Mass., (1979) 135-157, Lecture Notes in Pure and Appl. Math., **70**, Dekker, New York, (1981).

[44] N.G. Markley and M.E. Paul, *Matrix subshifts for Z^ν symbolic dynamics*, Proc. London Math. Soc., (3) **43**, (1981) 251-272.

[45] L. Onsager, *Crystal statistics. I. A two-dimensional model with an order-disorder transition.*, Phys. Rev. **65**, (1944) 117-149.

[46] C.N. Yang and C.P. Yang, *One-dimensional chain of anisotropic spin-spin interactions. I. Proof of Bethe's hypothesis for ground state in a finite system.*, Phys. Rev. **150**, (1966a) 321-327.

[47] C.N. Yang and C.P. Yang, *One-dimensional chain of anisotropic spin-spin interactions. II. Properties of the ground-state energy per lattice site for an infinite system.*, Phys. Rev. **150**, (1966b) 327-339.

[48] C.N. Yang and C.P. Yang, *One-dimensional chain of anisotropic spin-spin interactions. III. Applications.*, Phys. Rev. **151**, (1966c) 258-264.

Chapter 3

Patterns Generation and Spatial Entropy in Three-dimensional Lattice Models (I): Ordering Matrices and Connecting Operators

§ 3.1 Introduction

Lattice dynamical system(LDS) arise naturally in a wide applications of scientific models. See, for example, phase transitions [13], [14], [36], [37], [38], [39], [40], [47], [48], [49], [50], biology [10], [11], [23], [24], [25], [33], [34], [35], chemical reaction [8], [9], [26], image processing and pattern recognition [18], [19], [20], [21], [22], [27]. In cellular neural networks, much attention focus on the complexity of the set of all global patterns, in particular in its spatial entropy [1], [2], [3], [4], [5], [6], [7], [15], [16], [17], [30], [31], [32], [41], [42], [43], [44], [45], [46].

In a one-dimensional case, spatial entropy h can be exactly computed by a associated transition matrix $\mathbb{T}$, i.e., $h = \log \lambda(\mathbb{T})$, where $\lambda(\mathbb{T})$ is the maximum eigenvalue of $\mathbb{T}$.

For two-dimensional situations, [4] develops a systematical approach for discovering higher order transition matrix $\mathbb{T}_n$ and the spatial entropy h can be obtained by computing the maximum eigenvalues of a sequence of these transition matrices $\mathbb{T}_n$. For a class of admissible local patterns, i.e., for a class of $\mathbb{T}_2$, the limiting equation to $\rho^* = \exp(h(\mathbb{T}_2))$ can be exactly solved through the recursive formulae of $\rho(\mathbb{T}_n)$. However, $\mathbb{T}_n$ is a $2^n \times 2^n$ matrix, it is usually quite difficult to compute $\rho(\mathbb{T}_n)$ when n is larger. [5] derives the connecting operator to resolve these difficulties. Indeed, [5] yields lower-bound estimates of entropy by introducing connecting operators $\mathbb{C}_m$, and upper-bound estimates of entropy by introducing trace operators $\mathbb{T}_m$.

Our interest in this study is to develop a general approach for investigating three-dimensional pattern generation problems, i.e., extends works [4] and [5] to three-dimensional case. And this study focus on ordering matrices of patterns and connecting operator in three-dimensional case. The topic of trace operator will be appeared in [7].

More precisely, let $\mathcal{S}$ be a finite set of $p \geq 2$ colors where $\mathbb{Z}^3$ denotes the integer lattice of $\mathbb{R}^3$. Denote $U : \mathbb{Z}^3 \to \mathcal{S}$. And the set of all local patterns on $\mathbb{Z}_{m_1 \times m_2 \times m_3}$ is denoted by

$$\Sigma_{m_1 \times m_2 \times m_3} \equiv \{U|_{\mathbb{Z}_{m_1 \times m_2 \times m_3}} : U \in \Sigma_p^3\}$$

where $\mathbb{Z}_{m_1 \times m_2 \times m_3} = \{(\alpha_1, \alpha_2, \alpha_3) : \ 1 \leq \alpha_i \leq m_i, \ 1 \leq i \leq 3\}$ be a $m_1 \times m_2 \times m_3$ finite rectangular lattice. For simplicity, two colors on $2 \times 2 \times 2$ lattice $\mathbb{Z}_{2\times 2\times 2}$ are considered here. Given a basic set $\mathcal{B} \subset \Sigma_{2\times 2\times 2}$, the spatial entropy can be defined as

$$h(\mathcal{B}) = \lim_{m_1, m_2, m_3 \to \infty} \frac{\log \Gamma_{m_1 \times m_2 \times m_3}(\mathcal{B})}{m_1 m_2 m_3}, \tag{3.1.1}$$

where $\Gamma_{m_1 \times m_2 \times m_3}(\mathcal{B})$ is the number of distinct patterns in $\Sigma_{m_1 \times m_2 \times m_3}(\mathcal{B})$ and $\Sigma_{m_1 \times m_2 \times m_3}(\mathcal{B})$ is the set of all global patterns on $\mathbb{Z}_{m_1 \times m_2 \times m_3}(\mathcal{B})$ which can be generated by $\mathcal{B}$, as in [17]. Motivated by [4], there are six different orderings such as in (3.2.1) and according to the different ordering $[\omega]$ the ordering matrix $\mathbb{A}_{\omega;2\times 2\times 2}$ for $\Sigma_{2\times 2\times 2}$ can be introduced. Without loss of generality, we take the example $\mathbb{A}_{x;2\times 2\times 2}$ as in (3.2.9) and the other cases are similar. Use $[x]$-ordering on $\mathbb{Z}_{1\times m_2 \times 2}$ (3.2.26), the recursive formula of ordering matrix $\mathbb{A}_{x;2\times m_2\times 2}$ for $\Sigma_{2\times m_2 \times 2}$ can be obtained. Then, convert $[x]$-ordering into $[\hat{x}]$-ordering on $\mathbb{Z}_{1\times m_2\times 2}$ such as (3.2.27) enable introducing ordering matrix $\mathbb{A}_{\hat{x};2\times m_2\times 2}$ for $\Sigma_{2\times m_2\times 2}$. The recursive formulae of ordering matrix $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ for $\Sigma_{2\times m_2\times m_3}$ also be found through the $[\hat{x}]$-ordering on $\mathbb{Z}_{1\times m_2\times m_3}$ such as in (3.2.28). The recursive formula for $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ imply the recursive formula for the associated transition matrix $\mathbb{T}_{\hat{x};2\times m_2\times m_3}$ of $\Sigma_{2\times m_2\times m_3}(\mathcal{B})$ such as the Theorem 3.8 and Theorem 3.13, which enabling us to compute the maximum eigenvalue of $\mathbb{T}_{\hat{x};2\times m_2\times m_3}$ to get the spatial entropy such as in the Theorem 3.13. However, we hope to produce some estimations in spatial entropy $h(\mathcal{B})$. Then, for fixed m_1, $m_2 \leq 2$, the m_1-limit in (3.1.1) is studied, i.e.,

$$\lim_{m_3 \to \infty} \frac{1}{m_3} \log |\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times m_3}| \tag{3.1.2}$$

is considered. So the next task is to investigate of (3.1.2). As in (3.4.6) and (3.4.7), $A^{(k)}_{\hat{x};m_1,m_2,m_3;\alpha}$ is called an elementary pattern of order (m_1, m_2, m_3) and is a fundamental element in constructing $A^{(k)}_{\hat{x};m_1,m_2,m_3;\alpha}$ in (3.4.7). We define $\mathbb{X}_{\hat{x};m_1,m_2,m_3}$ as in (3.4.8) and (3.4.9) which is represented to reward systematically these elementary patterns. We introduce $\mathbb{C}_{\hat{x};m_3;m_1m_2}$ as 3.17, and use it to derive a recursive formulae for $A^{(k)}_{\hat{x};m_1,m_2,(m_3+1);\alpha_1;\alpha_2}$ and $A^{(\ell)}_{\hat{x};m_1,m_2,m_3;\alpha_2}$ as in 3.20.

The recursive formula (3.4.32) immediately yields a lower bound on entropy such as in 3.4.2. Equation (3.4.55) implies $h(\mathbb{A}_{x;2\times 2\times 2}) > 0$, if a diagonal periodic cycle applied, with a maximum eigenvalue in (3.4.55) larger than 1. This method powerfully yields the positivity of spatial entropy, which is hard in examining the the complexity of patterns generation problems.

The rest of this paper is organized as follows. Section 3.2, we derive a recursive formula to obtain the ordering matrix $\mathbb{A}_{x;2\times m_2\times 2}$ for $\Sigma_{2\times m_2\times 2}$ from $\mathbb{A}_{x;2\times 2\times 2}$. Convert the ordering $[x]$ into $[\hat{x}]$. Then, construct the similar recursive formula for ordering matrix $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ from $\mathbb{A}_{\hat{x};2\times m_2\times 2}$. Section 3.3 we derives the recursive formula for the associated higher order transition matrices $\mathbb{T}_{\hat{x};2\times m_2\times m_3}$ from $\mathbb{T}_{x;2\times 2\times 2}$. Section 3.4 derives the connecting operator $\mathbb{C}_m$ which can recursively reduce higher elementary patterns to patterns of lower order. Then, the lower-bound of spatial entropy can be found by computing the maximum eigenvalues of the diagonal periodic cycles of sequence $S_{\hat{x};m_3;m_1m}$.

§ 3.2 Three Dimensional Patterns Generation Problems

This section describes three dimensional patterns generation. Let $\mathcal{S}$ be a set of p colors, $\mathbb{Z}_{m_1\times m_2\times m_3}$ be a fixed finite rectangular sublattice of $\mathbb{Z}^3$, where $\mathbb{Z}^3$ denotes the integer lattice on $\mathbb{R}^3$ and (m_1, m_2, m_3) a 3-tuple of positive integers. Functions $U : \mathbb{Z}^3 \to \mathcal{S}$ and $U_{m_1\times m_2\times m_3} : \mathbb{Z}_{m_1\times m_2\times m_3} \to \mathcal{S}$ are called global patterns and local patterns on $\mathbb{Z}_{m_1\times m_2\times m_3}$ respectively. The set of all patterns U is denoted by $\Sigma_P \equiv \mathcal{S}^{\mathbb{Z}^3}$, i.e., Σ_p is the set of all patterns with p different colors in 3-dimensional lattice. For clarity, we begin by studying two symbols, i.e., $\mathcal{S} = \{0, 1\}$. There are three coordinates, let x-, y- and z-coordinate represent the 1st-, 2ed- and 3rd-coordinate respectively. There are six orderings $[\mathcal{O}]$ ordering could be represented as follows:

$$
\begin{array}{ccccccc}
[x] & : & [1] & \succ & [2] & \succ & [3], \\
[y] & : & [2] & \succ & [1] & \succ & [3], \\
[z] & : & [3] & \succ & [1] & \succ & [2], \\
[\hat{x}] & : & [1] & \succ & [3] & \succ & [2], \\
[\hat{y}] & : & [2] & \succ & [3] & \succ & [1], \\
[\hat{z}] & : & [3] & \succ & [2] & \succ & [1].
\end{array} \tag{3.2.1}
$$

On a fixed finite lattice $\mathbb{Z}_{m_1\times m_2\times m_3}$, we firstly give an ordering $[\mathcal{O}] = \mathcal{O}_{m_1\times m_2\times m_3}$ on $\mathbb{Z}_{m_1\times m_2\times m_3}$ which belongs to any one of above orderings on $\mathbb{Z}_{m_1\times m_2\times m_3}$ by $[\mathcal{O}] : [i] \succ [j] \succ [k]$

$$
\mathcal{O}(\alpha_1, \alpha_2, \alpha_3) = m_j m_k(\alpha_i - 1) + m_k(\alpha_j - 1) + \alpha_k. \tag{3.2.2}
$$

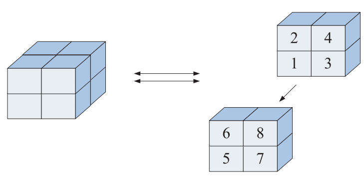
The ordering $[\mathcal{O}]$ on $\mathbb{Z}_{m_1\times m_2\times m_3}$ can now passed to $\Sigma_{m_1\times m_2\times m_3}$. Indeed for each $U=(u_{\alpha_1,\alpha_2,\alpha_3})\in\Sigma_{m_1\times m_2\times m_3}$, define

$$
\begin{aligned}
\mathcal{O}(U) &= \mathcal{O}_{m_1\times m_2\times m_3}(U) \\
&= 1+\sum_{\alpha_i=1}^{m_i}\sum_{\alpha_j=1}^{m_j}\sum_{\alpha_k=1}^{m_k} u_{\alpha_1\alpha_2\alpha_3}\mathcal{O}_{m_i,m_j,m_k}^{\alpha_i,\alpha_j,\alpha_k},
\end{aligned}
\tag{3.2.3}
$$

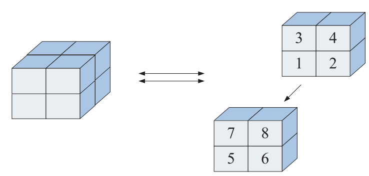
where

$$
\mathcal{O}_{m_i,m_j,m_k}^{\alpha_i,\alpha_j,\alpha_k} = 2^{m_k m_j(m_i-\alpha_i)+m_k(m_j-\alpha_j)+(m_k-\alpha_k)}. \tag{3.2.4}
$$

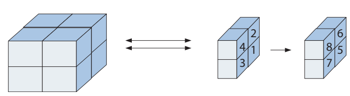
U is referred to herein as the $\mathcal{O}(U)$-th element in $\Sigma_{m_1\times m_2\times m_3}$ by ordering $[\mathcal{O}]$. By identifying the pictorial patterns by numbers $\mathcal{O}(U)$, it becomes highly effective in proving theorems since computations can now be performed on $\mathcal{O}(U)$. For example, the orderings on $\mathbb{Z}_{2\times 2\times 2}$ could be represented as follows:



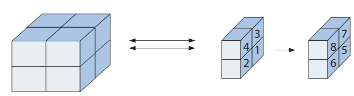
$[x]$-ordering



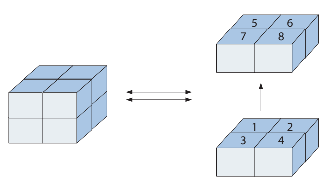
$[\hat{x}]$-ordering



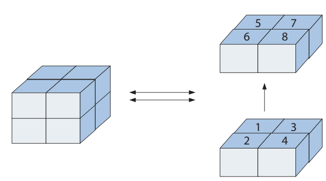
$[y]$-ordering



$[\hat{y}]$-ordering



$[z]$-ordering



$[\hat{z}]$-ordering

3.2.1 Ordering Matrices

Fixed $1\le\alpha_1\le m_1$, for $1\times m_2\times m_3$ pattern $U=(u_{\alpha_1\alpha_2\alpha_3})$, $1\le\alpha_2\le m_2$ and $1\le\alpha_3\le m_3$ in $\Sigma_{1\times m_2\times m_3}$, under the ordering $[x]$ pattern U is assigned the number

$$
i_{\alpha_1}=x(U)=1+\sum_{\alpha_2=1}^{m_2}\sum_{\alpha_3=1}^{m_3} u_{\alpha_1,\alpha_2,\alpha_3} x_{1,m_2,m_3}^{1,\alpha_2,\alpha_3}, \tag{3.2.5}
$$

where α_1 means the α_1-th layer in x-coordinate. As denoted by the $1 \times m_2 \times m_3$ pattern

$$(3.2.6)\quad a_{x;1\times m_2\times m_3;i_{\alpha_1}} = \begin{array}{|c|c|c|c|}\hline u_{\alpha_1 1 m_3} & u_{\alpha_1 2 m_3} & \cdots & u_{\alpha_1 m_2 m_3} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline u_{\alpha_1 12} & u_{\alpha_1 22} & \cdots & u_{\alpha_1 m_2 2} \\ \hline u_{\alpha_1 11} & u_{\alpha_1 21} & \cdots & u_{\alpha_1 m_2 1} \\ \hline \end{array}\ .$$

In particular, when $m_2 = 2$ and $m_3 = 2$ as denoted by $a_{x;1\times 2\times 2;i_{\alpha_1}}$, where

$$(3.2.7)\qquad i_{\alpha_1} = 1 + 2^3 u_{\alpha_1 11} + 2^2 u_{\alpha_1 12} + 2u_{\alpha_1 21} + u_{\alpha_1 22}$$

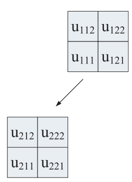
and

$$a_{x;1\times 2\times 2;i_{\alpha_1}} = \begin{array}{|c|c|}\hline u_{\alpha_1 12} & u_{\alpha_1 22} \\ \hline u_{\alpha_1 11} & u_{\alpha_1 21} \\ \hline \end{array}\ .$$

A $2 \times 2 \times 2$ pattern $U = (u_{\alpha_1\alpha_2\alpha_3})$ can now be obtained by $[x]$-direct sum of two $1 \times 2 \times 2$ patterns using $[x]$-ordering, i.e.,

$$a_{x;2\times 2\times 2;i_1 i_2} = a_{x;1\times 2\times 2;i_1} \bigoplus a_{x;1\times 2\times 2;i_2}$$

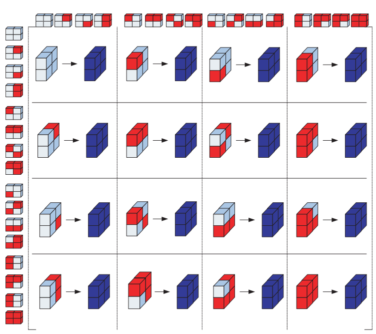
$$(3.2.8)\qquad =$$



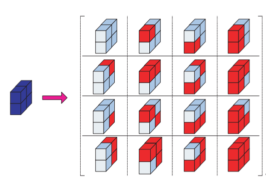
,

where i_{α_1} as in (3.2.7) and $\alpha_1 \in \{1, 2\}$. Therefore, the complete set of 2^8 patterns in $\Sigma_{2\times 2\times 2}$ can be listed by a 16×16 matrix $\mathbb{A}_{x;2\times 2\times 2} = [a_{x;2\times 2\times 2;i_1 i_2}]$ as its entries in

(3.2.9)



where


.

It is easy to verify that

$$(3.2.10)\qquad x(a_{x;2\times 2\times 2;i_1 i_2}) = 2^4(i_1 - 1) + i_2,$$

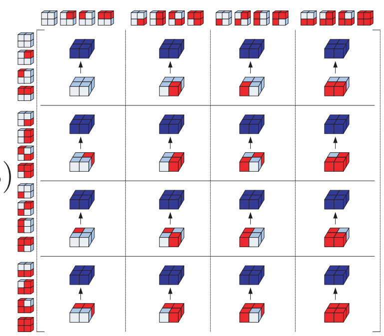
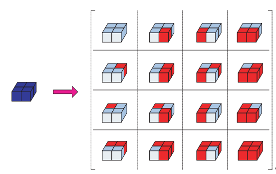
i.e., we are counting local patterns in $\Sigma_{2\times 2\times 2}$ by going through each row successively in (3.2.9). Correspondingly, $\mathbb{A}_{x;2\times 2\times 2}$ can be referred to as an ordering matrix for $\Sigma_{2\times 2\times 2}$. A $2\times 2\times 2$ pattern can also be viewed as $[x]$-direct sum of two $1\times 2\times 2$ patterns using $[\hat{x}]$-ordering, i.e.,

$$a_{\hat{x};2\times 2\times 2;\hat{i}_1\hat{i}_2} = a_{\hat{x};\hat{i}_1} \oplus a_{\hat{x};\hat{i}_2} \tag{3.2.11}$$

where

$$\hat{i}_{\alpha_1} = 1 + 2^3 u_{\alpha_1 11} + 2^2 u_{\alpha_1 21} + 2u_{\alpha_1 12} + u_{\alpha_1 22}, \alpha_1 \in \{1,2\}, \tag{3.2.12}$$

such as in (3.2.5). And the ordering matrix $\mathbb{A}_{\hat{x};2\times 2\times 2}$ can be represented as

(3.2.13)  where .

It could be verified that

$$\hat{x}(a_{\hat{x};\hat{i}_1\hat{i}_2}) = 2^4(\hat{i}_1 - 1) + \hat{i}_2. \tag{3.2.14}$$

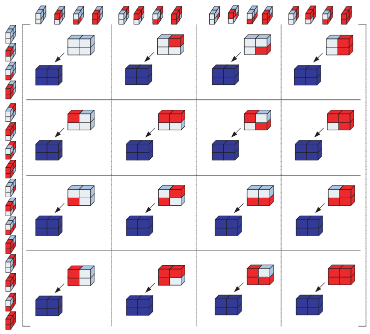
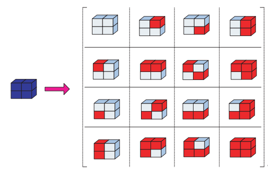
Similarly, a $2\times 2\times 2$ pattern can also be viewed as a $[y]$-direct ($[\hat{y}]$-direct) and $[z]$-direct ($[\hat{z}]$-direct) sum of $2\times 1\times 2$ and $2\times 2\times 1$ pattern, i.e.,

$$\begin{array}{rcccl} a_{y;j_1j_2} & = & a_{y;j_1} & \oplus & a_{y;j_2}, \\ a_{\hat{y};\hat{j}_1\hat{j}_2} & = & a_{\hat{y};\hat{j}_1} & \oplus & a_{\hat{y};\hat{j}_2}, \\ a_{z;k_1k_2} & = & a_{z;k_1} & \oplus & a_{z;k_2}, \\ a_{\hat{z};\hat{k}_1\hat{k}_2} & = & a_{\hat{z};\hat{k}_1} & \oplus & a_{\hat{z};\hat{k}_2}, \end{array}$$

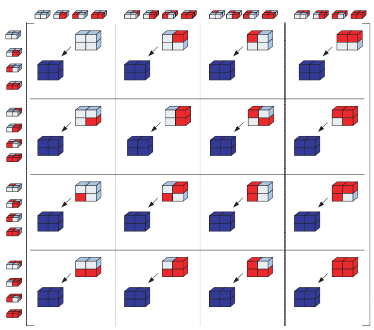
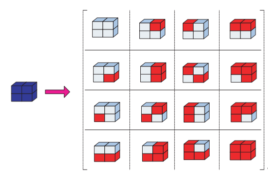
where

$$j_{\alpha_2} = 1 + 2^3 u_{1\alpha_2 1} + 2^2 u_{1\alpha_2 2} + 2u_{2\alpha_2 1} + u_{2\alpha_2 2},\ \alpha_2 \in \{1,2\}, \tag{3.2.15}$$
$$\hat{j}_{\alpha_2} = 1 + 2^3 u_{1\alpha_2 1} + 2^2 u_{2\alpha_2 1} + 2u_{1\alpha_2 2} + u_{2\alpha_2 2},\ \alpha_2 \in \{1,2\}, \tag{3.2.16}$$
$$k_{\alpha_3} = 1 + 2^3 u_{11\alpha_3} + 2^2 u_{12\alpha_3} + 2u_{21\alpha_3} + u_{22\alpha_3},\ \alpha_3 \in \{1,2\}, \tag{3.2.17}$$
$$\hat{k}_{\alpha_3} = 1 + 2^3 u_{11\alpha_3} + 2^2 u_{21\alpha_3} + 2u_{12\alpha_3} + u_{22\alpha_3},\ \alpha_3 \in \{1,2\}. \tag{3.2.18}$$

A 16×16 matrix $\mathbb{A}_{y;2\times2\times2} = [a_{y;2\times2\times2;j_1j_2}]$ or $\mathbb{A}_{z;2\times2\times2} = [a_{z;2\times2\times2;k_1k_2}]$ can also be obtained for $\Sigma_{2\times2\times2}$, i.e., we have $\mathbb{A}_{y;2\times2\times2} =$

(3.2.19)  where ,

or $\mathbb{A}_{z;2\times2\times2}$

(3.2.20)  where .

The relations between $\mathbb{A}_{\omega;2\times2\times2}$ must be explored, where $\omega \in \{x, y, z, \hat{x}, \hat{y}, \hat{z}\}$. Before explaining the relations we denote column matrix and row matrix. Let $\mathbb{A} = [a_{ij}]$ be a $m^2 \times m^2$ matrix, the column matrix $\mathbb{A}^{(c)}$ of $\mathbb{A}$ is defined by

$$\mathbb{A}^{(c)} = \begin{bmatrix} A_1^{(c)} & A_2^{(c)} & \cdots & A_{m^2}^{(c)} \\ A_{m^2+1}^{(c)} & A_{m^2+2}^{(c)} & \cdots & A_{2m^2}^{(c)} \\ \vdots & \vdots & \ddots & \vdots \\ A_{(m^2-1)m^2+1}^{(c)} & A_{(m^2-1)m^2+2}^{(c)} & \cdots & A_{m^4}^{(c)} \end{bmatrix}, \tag{3.2.21}$$

$$A_{alpha}^{(c)} = \begin{bmatrix} a_{1\alpha} & a_{2\alpha} & \cdots & a_{m^2\alpha} \\ a_{(m^2+1)\alpha} & a_{(m^2+2)\alpha} & \cdots & a_{(2m^2)\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ a_{((m^2-1)m^2+1)\alpha} & a_{((m^2-1)m^2+2)\alpha} & \cdots & a_{m^4\alpha} \end{bmatrix}, \tag{3.2.22}$$

where$1 \leq \alpha \leq m^4$.

And the row matrix $\mathbb{A}^{(r)}$ of $\mathbb{A}$ is defined by

$$\mathbb{A}^{(r)} = \tag{3.2.23}$$
$$\begin{bmatrix} A_1^{(r)} & A_2^{(r)} & \cdots & A_{m^2}^{(r)} \\ A_{m^2+1}^{(r)} & A_{m^2+2}^{(r)} & \cdots & A_{2m^2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ A_{(m^2-1)m^2+1}^{(r)} & A_{(m^2-1)m^2+2}^{(r)} & \cdots & A_{m^4}^{(r)} \end{bmatrix},$$

$$A_\alpha^{(r)} = \tag{3.2.24}$$
$$\begin{bmatrix} a_{\alpha 1} & a_{\alpha 2} & \cdots & a_{\alpha m^2} \\ a_{\alpha(m^2+1)} & a_{\alpha(m^2+2)} & \cdots & a_{\alpha(2m^2)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\alpha((m^2-1)m^2+1)} & a_{\alpha((m^2-1)m^2+2)} & \cdots & a_{\alpha m^4} \end{bmatrix},$$

where $1 \leq \alpha \leq m^4$. Therefore, from some observations, $\mathbb{A}_{x;2\times 2\times 2}$ can be represented by $a_{y;j_1j_2}$ as

$$\mathbb{A}_{x;2\times 2\times 2} = \mathbb{A}_{y;2\times 2\times 2}^{(r)}. \tag{3.2.25}$$

The remainder of this subsection is devoted to construct $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ from $\mathbb{A}_{x;2\times 2\times 2}$ by the following three steps, where $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ represented the ordering matrix of $\Sigma_{2\times m_2\times m_3}$ according to $[\hat{x}]$-ordering generated from $\Sigma_{2\times 2\times 2}$.

Step I : Use $[x]$-ordering on $\mathbb{Z}_{1\times m_2\times 2}$ by

(3.2.26)

2	4	...	2k	...	$2m_2$-2	$2m_2$
1	3	...	2k-1	...	$2m_2$-3	$2m_2$-1

y

and introduce ordering matrix $\mathbb{A}_{x;2\times m_2\times 2}$ for $\Sigma_{2\times m_2\times 2}$.

Step II : Convert $[x]$-ordering into $[\hat{x}]$-ordering on $\mathbb{Z}_{1\times m_2\times 2}$ by

(3.2.27)

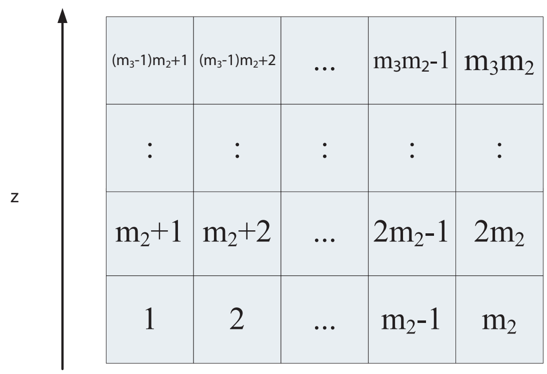
z

m_2+1	m_2+2	...	m_2+k	...	$2m_2$
1	2	...	k	...	m_2

and introduce ordering matrix $\mathbb{A}_{\hat{x};2\times m_2\times 2}$ for $\Sigma_{2\times m_2\times 2}$.

Step III : Define $[\hat{x}]$-ordering on $\mathbb{Z}_{1\times m_2\times m_3}$ by

(3.2.28)



and introduce ordering matrix $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ for $\Sigma_{2\times m_2\times m_3}$.

To introduce $\mathbb{A}_{x;2\times m_2\times 2}$, define

$$\begin{aligned} a_{y;2\times m_2\times 2;j_1j_2\ldots j_{2m_2}} &= a_{y;2\times2\times2;j_1j_2}\hat{\bigoplus} a_{y;2\times2\times2;j_2j_3}\hat{\bigoplus}\cdots\hat{\bigoplus} a_{y;2\times2\times2;j_{m_2-1}j_{m_2}} \\ &= a_{y;j_1}\oplus a_{y;j_2}\oplus\cdots\oplus a_{y;j_{m_2}}, \end{aligned} \tag{3.2.29}$$

where $1\le j_k\le 2^4$ and $1\le k\le m_2$. Herein, a wedge direct sum $\hat{\bigoplus}$ is used for $2\times2\times2$ patterns whenever they can attached together.

Now, $A_{x;2\times m_2\times 2}$ can be obtained as follows.

Theorem 3.1. For any $m_2\ge 2$, $\Sigma_{2\times m_2\times 2}=\{a_{y;j_1j_2\ldots j_{m_2}}\}$, where $a_{y;j_1j_2\ldots j_{2m_2}}$ is given in (3.2.29). Furthermore, the ordering matrix $\mathbb{A}_{x;2\times m_2\times 2}=[a_{y;j_1j_2\cdots j_{m_2}}]$ which is a $2^{2m_2}\times 2^{2m_2}$ matrix can be decomposed into following matrices

$$\mathbb{A}_{x;2\times m_2\times 2}=[A_{x;2\times m_2\times 2;j_1}]_{2^{m_2}\times 2^{m_2}},$$

where $1\le j_1\le 2^{2m_2}$. For fixed $j_1,j_2,\ldots,j_k\in\{1,2,\ldots,2^{2m_2}\}$,

$$A_{x;2\times m_2\times 2;j_1j_2\ldots j_k}=[A_{x;2\times m_2\times 2;j_1j_2\ldots j_kj_{k+1}}]_{m_2\times m_2},$$

where $1\le j_{k+1}\le 2^{2m_2}$ and $k\in\{1,2,\cdots,m_2-2\}$. For fixed $j_1,j_2,\cdots,j_{m_2-1}$,

$$A_{x;2\times m_2\times 2;j_1j_2\ldots j_{m_2-1}}=[a_{y;2\times m_2\times 2;j_1j_2\ldots j_{m_2-1}j_{m_2}}]_{2^{m_2}\times 2^{m_2}},$$

where $a_{y;2\times m_2\times 2;j_1j_2\ldots j_{m_2}}$ is defined in (3.2.29).

Proof. From (3.2.15), $u_{\alpha_1\alpha_2\alpha_3}$ can be solved in terms of j_{α_2}, i.e., we have

$$u_{1\alpha_21} = [\frac{j_{\alpha_2}-1}{2^3}], \tag{3.2.30}$$

$$u_{1\alpha_22} = [\frac{j_{\alpha_2}-1-2^3u_{1\alpha_21}}{2^2}], \tag{3.2.31}$$

$$u_{2\alpha_21} = [\frac{j_{\alpha_2}-1-2^3u_{1\alpha_21}-2^2u_{1\alpha_22}}{2}], \tag{3.2.32}$$

$$u_{2\alpha_22} = j_{\alpha_2}-1-2^3u_{1\alpha_21}-2^2u_{1\alpha_22}-2u_{2\alpha_21}, \tag{3.2.33}$$

where [ ] is the Gauss symbol. From (3.2.30) to (3.2.33), we have the following table.

j_{α_2}	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$u_{1\alpha_2 1}$	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
$u_{1\alpha_2 2}$	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
$u_{2\alpha_2 1}$	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
$u_{2\alpha_2 2}$	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

For any $m_2 \geq 2$, we have

$$i_{m_2;1} = 1 + \sum_{\alpha_2=1}^{m_2} (2^{2(m_2-\alpha_2)+1} u_{1\alpha_2 1} + 2^{2(m_2-\alpha_2)} u_{1\alpha_2 2}), \tag{3.2.34}$$

$$i_{m_2;2} = 1 + \sum_{\alpha_2=1}^{m_2} (2^{2(m_2-\alpha_2)+1} u_{2\alpha_2 1} + 2^{2(m_2-\alpha_2)} u_{2\alpha_2 2}). \tag{3.2.35}$$

From above formulae, we have

$$\begin{aligned} i_{m_2+1;1} &= 2^2(i_{m_2;1} - 1) + 2u_{1(m_2+1)1} + u_{1(m_2+1)2} + 1, \\ i_{m_2+1;2} &= 2^2(i_{m_2;2} - 1) + 2u_{2(m_2+1)1} + u_{2(m_2+1)2} + 1. \end{aligned}$$

Now, by induction on m_2 the theorem follows from last two formulae and the above table. The proof is complete. $\square$

Remark 3.2. By the similar method, the following relations ca be derived but the detailed proof is omitted here for brevity.

$$A_{\hat{x};2\times 2\times m_3} = [a_{z;2\times 2\times m_3;k_1k_2\dots k_{m_3-1}k_{m_3}}]_{2^{m_3}\times 2^{m_3}} \tag{3.2.36}$$

$$A_{y;m_1\times 2\times 2} = [a_{x;m_1\times 2\times 2;i_1i_2\dots i_{m_1-1}i_{m_1}}]_{2^{m_1}\times 2^{m_1}} \tag{3.2.37}$$

$$A_{\hat{y};2\times 2\times m_3} = [a_{\hat{z};2\times 2\times m_3;k_1k_2\dots k_{m_3-1}k_{m_3}}]_{2^{m_3}\times 2^{m_3}} \tag{3.2.38}$$

$$A_{z;m_1\times 2\times 2} = [a_{\hat{x};m_1\times 2\times 2;i_1i_2\dots i_{m_1-1}i_{m_1}}]_{2^{m_1}\times 2^{m_1}} \tag{3.2.39}$$

$$A_{\hat{z};2\times m_2\times 2} = [a_{\hat{y};2\times m_2\times 2;j_1j_2\dots j_{m_2-1}j_{m_2}}]_{2^{m_2}\times 2^{m_2}} \tag{3.2.40}$$

Next, $[x]$-ordering is converted into $[\hat{x}]$-ordering for $\mathbb{Z}_{1\times m_2\times 2}$. Since $\mathbb{Z}_{1\times m_2\times 2} = \{(1,\alpha_2,\alpha_3) : 1 \leq \alpha_2 \leq m_2, 1 \leq \alpha_3 \leq 2\}$, the position (α_2,α_3) is the α-th in (3.2.26), where

$$\alpha = 2(\alpha_2 - 1) + \alpha_3. \tag{3.2.41}$$

In (3.2.27), the position of $(1,\alpha_2,\alpha_3)$ is the $\hat{\alpha}$-th, where

$$\hat{\alpha} = m_2(\alpha_3 - 1) + \alpha_2. \tag{3.2.42}$$

It is easy to verify

$$\hat{\alpha} = m_2\alpha + (1-2m_2)[\frac{\alpha-1}{2}] + (1-m_2), \tag{3.2.43}$$

or

$$\hat{\alpha} = k \qquad if \quad \alpha = 2k-1,$$

and

$$\hat{\alpha} = m_2 + k \qquad if \quad \alpha = 2k,$$

$1 \leq k \leq m_2$.

Now, the ordering $[\hat{x}]$ in (3.2.27) on $\mathbb{Z}_{1\times m_2\times 2}$ can be extended to $\mathbb{Z}_{1\times m_2\times m_3}$ by (3.2.28). For a fixed m_2, $[\hat{x}]$-ordering on $\mathbb{Z}_{1\times m_2\times m_3}$ is clearly one dimensional; it grows in z-direction. With ordering (3.2.28) on $\mathbb{Z}_{1\times m_2\times m_3}$, for $U = (u_{\alpha_1\alpha_2\alpha_3}) \in \Sigma_{2\times m_2\times m_3}$, denoted by

$$\hat{i}_{\alpha_1} = 1 + \sum_{\alpha_2=1}^{m_2}\sum_{\alpha_3=1}^{m_3} u_{\alpha_1\alpha_2\alpha_3} 2^{m_2(m_3-\alpha_3)+(m_2-\alpha_2)}, \tag{3.2.44}$$

where $\alpha_1 = 1, 2$. Then, we obtain

$$\hat{x}(U) = 2^{m_2m_3}(\hat{i_1}-1) + \hat{i_2}. \tag{3.2.45}$$

Now, let $a_{\hat{x};\hat{i_1}\hat{i_2}} = U = (u_{\alpha_1\alpha_2\alpha_3})$, then we have new ordering matrix $\mathbb{A}_{\hat{x};2\times m_2\times 2} = [a_{\hat{x};2\times m_2\times 2;\hat{i_1}\hat{i_2}}]$ for $\Sigma_{2\times m_2\times 2}$. The relationship between $\mathbb{A}_{x;2\times m_2\times 2}$ and $\mathbb{A}_{\hat{x};2\times m_2\times 2}$ is established before constructing $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ from $\mathbb{A}_{\hat{x};2\times m_2\times 2}$ for $m_3 \geq 3$.

We firstly established a conversion sequence of orderings from (3.2.26) to (3.2.27). Where P_k denotes the permutation of $\mathbb{N}_{2m} = \{1, 2, \cdots, 2m_2\}$ such that $P_k(k+1) = k$,$P_k(k) = k+1$ and the other numbers are fixed. We also denote P_k the permutation on $\mathbb{Z}_{1\times m_2\times 2}$ such that it exchanges k and k+1 and maintains the other positions fixed, i.e,

$$\begin{array}{|c|c|c|c|c|c|c|}\hline \cdot & & k+1 & & \cdot & & \cdot \\ \hline \cdot & & \cdot & & k & & \cdot \\ \hline \end{array} \xrightarrow{P_k} \begin{array}{|c|c|c|c|c|c|c|}\hline \cdot & & k & & \cdot & & \cdot \\ \hline \cdot & & \cdot & & k+1 & & \cdot \\ \hline \end{array} \tag{3.2.46}$$

Obviously (3.2.26) can be converted into (3.2.27) in many ways by using sequence of P_k. Here, we present a systematic approach.

Lemma 3.3. For $m_2 \geq 2$, (3.2.26) can be converted into (3.2.27) by the following sequences of $\frac{m_2(m_2-1)}{2}$ permutations successively

$$\begin{array}{c}(P_2P_4\cdots P_{2m_2-2})(P_3P_5\cdots P_{2m_2-3})\cdots \\ (P_kP_{k+2}\cdots P_{2m_2-k})\cdots(P_{m_2-1}P_{m_2+1})P_{m_2},\end{array} \tag{3.2.47}$$

$2 \leq k \leq m_2$.

Proof. When $m_2 = 2$ and 3, verifying that (3.2.47) can convert (3.2.26) into (3.2.27) is relatively easy.

When $m_2 \geq 4$, and for any $2 \leq k \leq m_2$, applying

$$(3.2.48)\quad (P_2 P_4 \cdots P_{2m_2-2})(P_3 P_5 \cdots P_{2m_2-3}) \cdots (P_k P_{k+2} \cdots P_{2m_2-k})$$

to (3.2.26), then there are two intermediate cases:
(i) when $2 \leq k \leq [\frac{m_2}{2}]$, then we have

(3.2.49)

$k+1$	$k+3$	$\cdots$	$3k-1$	$\cdots$	$\cdots$	$\cdots$	$3k-1+2\ell$	$\cdots$	$2m_2-k-1$	$2m_2-k+1$	$\cdots$	$2m_2-1$	$2m_2$
1	2	$\cdots$	k	$k+2$	$k+4$	$\cdots$	$k+2\ell$	$\cdots$	$\cdots$	$2m_2-3k+1$	$\cdots$	$2m_2-k-2$	$2m_2-k$

where $0 \leq \ell \leq m_2 - 2k$.
(ii) when $[\frac{m_2}{2}] + 1 \leq k \leq m_2 - 1$, then we have

(3.2.50)

$k+1$	$\cdots$	$2m_2-k-1$	$2m_2-k+1$	$2m_2-k+2$	$\cdots$	$\cdots$	$\cdots$	$2m_2-1$	$2m_2$
1	2	$\cdots$	$\cdots$	$\cdots$	$k-1$	k	$k+2$	$\cdots$	$2m_2-k$

When $k = m_2$ in (3.2.50), we have (3.2.27). We prove (3.2.49) and (3.2.50) by mathematical induction on k. When k=2, it is relatively easy to verify that (3.2.26) is converted into

3	5	$\cdots$	$\cdots$	$\cdots$	$2m_2-3$	$2m_2-1$	$2m_2$
1	2	4	$\cdots$	$\cdots$	$\cdots$	$2m_2-1$	$2m_2-2$

by $P_2 P_4 \cdots P_{2m_2-2}$, i.e., (3.2.49) holds for k=2. Next, assume that (3.2.49) holds for $k \leq [\frac{m}{2}]$. Then, by applying $P_{k+1} P_{k+2} \cdots P_{2m_2-k-1}$ to (3.2.49), it can be verified that (3.2.49) holds for $k+1$ when $k+1 \leq [\frac{m_2}{2}]$ or becomes (3.2.50) when $k+1 \geq [\frac{m_2}{2}]$. When $k \geq [\frac{m_2}{2}] + 1$, we apply $P_{k+1} P_{k+3} \cdots P_{2m_2-k-1}$ to (3.2.50). It can also be verified that (3.2.50) holds for k+1. Finally, we conclude that (4.27) holds for $k = m_2$. The proof is thus complete. $\square$

By using Lemma 3.3, $\mathbb{A}_{x;2\times m_2\times 2}$ can be converted into $\mathbb{A}_{\hat{x};2\times m_2\times 2}$ by the following construction. Let

$$(3.2.51)\qquad P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and for $2 \le j \le 2m_2 - 2$, as denoted by

$$P_{x;2m_2;j} = I_{2^{j-1}} \otimes P \otimes I_{2^{2m_2-j-1}}, \tag{3.2.52}$$

where I_k is the $k \times k$ identity matrix. Furthermore, let

$$\begin{aligned} \mathbb{P}_{x;2\times m_2\times 2} &= (P_{2m_2;2}P_{2m_2;4}\cdots P_{2m_2;2m_2-2}) \\ &\quad \cdots(P_{2m_2;k}\cdots P_{2m_2;2m_2-k})\cdots(P_{2m_2;m_2}), \end{aligned} \tag{3.2.53}$$

$2 \le k \le m_2$. Then, we have the following theorem.

Theorem 3.4. For any $m_2 \ge 2$,

$$\mathbb{A}_{\hat{x};2\times m_2\times 2} = \mathbb{P}^t_{x;2\times m_2\times 2}\mathbb{A}_{x;2\times m_2\times 2}\mathbb{P}_{x;2\times m_2\times 2}. \tag{3.2.54}$$

Proof. From (3.2.41), in $\mathbb{Z}_{1\times m_2\times 2}$ the position (α_2, α_3) is the α-th in (3.2.26), where $\alpha = 2(\alpha_2 - 1) + \alpha_3$. Define

$$\ell_\alpha = 1 + 2u_{1\alpha_2\alpha_3} + u_{2\alpha_2\alpha_3}, \tag{3.2.55}$$

$1 \le \ell_\alpha \le 4$ and $1 \le \alpha \le 2m_2$. For $U = (u_{\alpha_1\alpha_2\alpha_3}) \in \Sigma_{2\times m_2\times 2}$, from Theorem 3.1 it can be denoted by $a_{y;2\times m_2\times 2;j_1j_2\cdots j_{m_2}}$ and by (3.2.15) for fixed $1 \le \alpha_2 \le m_2$ we have

$$\begin{aligned} j_{\alpha_2} &= 1 + 2^3u_{1\alpha_21} + 2^2u_{1\alpha_22} + 2u_{2\alpha_21} + u_{2\alpha_22} \\ &= 2^2(\ell_{2\alpha_2-1}) + \ell_{2\alpha_2} + 1, \end{aligned}$$

where $1 \le j_{\alpha_2 \le 16}$. Hence the relation between $a_{y;j_{\alpha_2}}$ and $w_{y;\ell_{2\alpha_2-1}\ell_{2\alpha_2}}$ is

$$\begin{bmatrix} a_{y;1} & a_{y;2} & a_{y;3} & a_{y;4} \\ a_{y;5} & a_{y;6} & a_{y;7} & a_{y;8} \\ a_{y;9} & a_{y;10} & a_{y;11} & a_{y;12} \\ a_{y;13} & a_{y;14} & a_{y;15} & a_{y;16} \end{bmatrix} = \begin{bmatrix} w_{11} & w_{12} & w_{21} & w_{22} \\ w_{13} & w_{14} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{41} & w_{42} \\ w_{33} & w_{34} & w_{43} & w_{44} \end{bmatrix}.$$

Therefore, the pattern in ordering matrix $\mathbb{A}_{x;2\times m_2\times 2}$ can be represented by

$$\begin{aligned} a_{y;2\times m_2\times 2;j_1j_2\ldots j_{m_2}} &= a_{y;j_1} \oplus a_{y;j_2} \oplus \cdots \oplus a_{y;j_{m_2}} \\ &= w_{y;\ell_1\ell_2} \oplus w_{y;\ell_3\ell_4} \oplus \cdots \oplus w_{y;\ell_{2m_2-1}\ell_{2m_2}} \\ &\equiv w_{y;\ell_1\ell_2\ldots\ell_{2m_2}}. \end{aligned}$$

It is easy to verify that for any $1 \le k \le 2m_2 - 1$,

$$\begin{aligned} &P^t_{2m_2;k}\mathbb{A}_{x;2\times m_2\times 2}P_{2m_2;k} \\ &= P^t_{2m_2;k}[w_{y;\ell_1\ell_2\ldots\ell_k\ell_{k+1}\ldots\ell_{2m_2}}]P_{2m_2;k} \\ &= [w_{y;\ell_1\ell_2\ldots\ell_{k+1}\ell_k\ldots\ell_{2m_2}}], \end{aligned}$$

i.e., $P_{2m_2;k}$ exchanges ℓ_k and ℓ_{k+1} in $\mathbb{A}_{x;2\times m_2\times 2}$. Therefore, from (3.2.53) and Lemma 3.3, (3.2.54) follows. ∎

Now, in Theorem 3.4, as denoted by

$$\mathbb{A}_{\hat{x};2\times m_2\times 2} = [a_{\hat{x};2\times m_2\times 2;\hat{i_1}\hat{i_2}}], \tag{3.2.56}$$

$1 \leq \hat{i}_1, \hat{i}_2 \leq 2m_2$. And by Remark 3.2, $\mathbb{A}_{\hat{x};2\times m_2\times 2}$ could be represented by $a_{z;2\times m_2\times 2;k_1k_2}$, where $1 \leq k_1, k_2 \leq 2^{2m_2}$. The $[\hat{x}]$-expression

$$\mathbb{A}_{\hat{x};2\times m_2\times 2} = \mathbb{A}^{(r)}_{z;2\times m_2\times 2} \tag{3.2.57}$$

for $\Sigma_{2\times m_2\times 2}$ enable us to construct $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ for $\Sigma_{2\times m_2\times m_3}$. Indeed, for fixed $m_2 \geq 2$ and $m_3 \geq 2$, let

$$\begin{aligned} a_{\hat{x};2\times m_2\times m_3;\hat{i_1}\hat{i_2}} &= a_{z;2\times m_2\times m_3;k_1k_2\ldots k_{m_3}} \\ &= a_{z;2\times m_2\times 2;k_1k_2}\hat{\oplus}a_{z;2\times m_2\times 2;k_2k_3}\hat{\oplus}\cdots\hat{\oplus}a_{z;2\times m_2\times 2;k_{m_3-1}k_{m_3}}. \end{aligned} \tag{3.2.58}$$

Therefore, by a similar argument as in proving Theorem 3.1 we have the following theorem for $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$. The detailed proof is omitted here for brevity.

Theorem 3.5. By fixing $m_2 \geq 2$ and for any $m_3 \geq 2$, the ordering matrix $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ with respect to $[\hat{x}]$-ordering can be expressed as

$$\mathbb{A}_{\hat{x};2\times m_2\times m_3} = [A_{\hat{x};2\times m_2\times m_3;k_1}]_{2^{m_2}\times 2^{m_2}}, \tag{3.2.59}$$

where $1 \leq k_1 \leq 2^{2m_2}$. For fixed $1 \leq k_1, k_2, \cdots, k_l \leq 2^{2m_2}$,

$$A_{\hat{x};2\times m_2\times m_3;k_1k_2\cdots k_l} = [A_{\hat{x};2\times m_2\times m_3;k_1k_2\cdots k_lk_{l+1}}]_{2^{m_2}\times 2^{m_2}} \tag{3.2.60}$$

where $1 \leq k_{l+1} \leq 2^{2m_2}$ and $1 \leq l \leq m_3 - 2$. For fixed $k_1, k_2, \cdots, k_{m_3-1}$,

$$A_{\hat{x};2\times m_2\times m_3;k_1k_2\cdots k_{m_3-1}} = [a_{z;2\times m_2\times m_3;k_1k_2\ldots k_{m_3}}], \tag{3.2.61}$$

where $a_{z;2\times m_2\times m_3;k_1k_2\ldots k_{m_3}}$ is given by (3.2.58).

Remark 3.6. Similarly, the following relations can be derived but the detailed proof is omitted here for brevity.

$$\begin{aligned} A_{x;2\times m_2\times m_3} &= [a_{y;2\times m_2\times m_3;j_1j_2\ldots j_{m_2}}]_{2^{m_2m_3}\times 2^{m_2m_3}} \\ A_{\hat{y};m_1\times 2\times m_3} &= [a_{\hat{z};m_1\times 2\times m_3;k_1k_2\ldots k_{m_3}}]_{2^{m_1m_3}\times 2^{m_1m_3}} \\ A_{y;m_1\times 2\times m_3} &= [a_{x;m_1\times 2\times m_3;i_1i_2\ldots i_{m_1}}]_{2^{m_1m_3}\times 2^{m_1m_3}} \\ A_{\hat{z};m_1\times m_2\times 2} &= [a_{\hat{y};m_1\times m_2\times 2;j_1j_2\ldots j_{m_2}}]_{2^{m_1m_2}\times 2^{m_1m_2}} \\ A_{z;m_1\times m_2\times 2} &= [a_{\hat{x};m_1\times m_2\times 2;i_1i_2\ldots i_{m_1}}]_{2^{m_1m_2}\times 2^{m_1m_2}} \end{aligned}$$

§ 3.3 Transition Matrices and Spatial Entropy

3.3.1 Transition Matrices

With the ordering matrices $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ for $\Sigma_{2\times m_2\times m_3}$ having been defined, higher order transition matrices $\mathbb{T}_{\hat{x};2\times m_2\times m_3}$ can now be derived from $\mathbb{T}_{x;2\times 2\times 2}$. As in the two dimensional case [4], assume that we have basic set $\mathcal{B}\subset\Sigma_{2\times 2\times 2}$. Define the transition matrix $\mathbb{T}_{x;2\times 2\times 2}=\mathbb{T}_{x;2\times 2\times 2}(\mathcal{B})$ by

$$\mathbb{T}_{x;2\times 2\times 2}=[t_{x;2\times 2\times 2;i_1 i_2}]_{2^4\times 2^4}, \tag{3.3.1}$$

where

$$\begin{aligned} t_{x;2\times 2\times 2;i_1 i_2} &= 1 \quad if \quad a_{x;2\times 2\times 2;i_1 i_2}\in\mathcal{B}, \\ &= 0 \qquad\quad otherwise. \end{aligned} \tag{3.3.2}$$

Then, the transition matrix $\mathbb{T}_{x;2\times m_2\times 2}$ is a $2^{2m_2}\times 2^{2m_2}$ matrix with entries $t_{x;2\times m_2\times 2;i_1 i_2}$, where

$$\begin{aligned} t_{x;2\times m_2\times 2;i_1 i_2} &= t_{y;2\times m_2\times 2;j_1 j_2\ldots j_{m_2}} \\ &= \prod_{k=1}^{m_2-1} t_{y;2\times 2\times 2;j_k j_{k+1}}. \end{aligned} \tag{3.3.3}$$

Before $\mathbb{T}_{x;2\times m_2\times 2}$ is introduced, three products of matrices are defined as follows.

Definition 3.7. For any two matrices $\mathbb{M}=(M_{ij})$ and $\mathbb{N}=(N_{kl})$, the Kronecker product (tensor product) $\mathbb{M}\otimes\mathbb{N}$ of $\mathbb{M}$ and $\mathbb{N}$ is defined by

$$\mathbb{M}\otimes\mathbb{N}=(M_{ij}\mathbb{N}). \tag{3.3.4}$$

For any $n\geq 1$,

$$\otimes\mathbb{N}^n=\mathbb{N}\otimes\mathbb{N}\otimes\cdots\otimes\mathbb{N},$$

n-times in $\mathbb{N}$.
Next, for any two $m\times m$ matrices

$$\mathbb{P}=(P_{ij})\ and\ \mathbb{Q}=(Q_{ij})$$

where P_{ij} and Q_{ij} are numbers or matrices, the Hadamard product $\mathbb{P}\circ\mathbb{Q}$ is defined by

$$\mathbb{P}\circ\mathbb{Q}=(P_{ij}\cdot Q_{ij}), \tag{3.3.5}$$

where the product $P_{ij} \cdot Q_{ij}$ of P_{ij} and Q_{ij} may be a multiplication between numbers, between numbers and matrices or between matrices whenever it is well-defined.

Finally, product $\hat{\otimes}$ is defined as follows. For any 4×4 matrix

$$\mathbb{M}_2 = \begin{bmatrix} m_{11} & m_{12} & m_{21} & m_{22} \\ m_{13} & m_{14} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{41} & m_{42} \\ m_{33} & m_{34} & m_{43} & m_{44} \end{bmatrix} = \begin{bmatrix} M_{2;1} & M_{2;2} \\ M_{2;3} & M_{2;4} \end{bmatrix} \tag{3.3.6}$$

and any 2×2 matrix

$$\mathbb{N} = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix}, \tag{3.3.7}$$

where m_{ij} are numbers and N_k are numbers or matrices, for $1 \le i, j, k \le 4$, define

$$\mathbb{M}_2 \hat{\otimes} \mathbb{N} = \begin{bmatrix} m_{11}N_1 & m_{12}N_2 & m_{21}N_1 & m_{22}N_2 \\ m_{13}N_3 & m_{14}N_4 & m_{23}N_3 & m_{24}N_4 \\ m_{31}N_1 & m_{32}N_2 & m_{41}N_1 & m_{42}N_2 \\ m_{33}N_3 & m_{34}N_4 & m_{43}N_3 & m_{44}N_4 \end{bmatrix}. \tag{3.3.8}$$

Furthermore, for $n \ge 1$, the $n+1$ th order of transition matrix of $\mathbb{M}_2$ is defined by

$$\mathbb{M}_{n+1} \equiv \hat{\otimes}\mathbb{M}_2^n = \mathbb{M}_2 \hat{\otimes} \mathbb{M}_2 \hat{\otimes} \cdots \hat{\otimes} \mathbb{M}_2,$$

n-times in $\mathbb{M}_2$. More precisely,

$$\mathbb{M}_{n+1} = \mathbb{M}_2 \hat{\otimes} (\hat{\otimes}\mathbb{M}_2^{n-1}) = \begin{bmatrix} M_{2;1} \circ (\hat{\otimes}\mathbb{M}_2^{n-1}) & M_{2;2} \circ (\hat{\otimes}\mathbb{M}_2^{n-1}) \\ M_{2;3} \circ (\hat{\otimes}\mathbb{M}_2^{n-1}) & M_{2;4} \circ (\hat{\otimes}\mathbb{M}_2^{n-1}) \end{bmatrix}$$

(3.3.9)

$$= \left[\begin{array}{cc|cc} m_{11}M_{n;1} & m_{12}M_{n;2} & m_{21}M_{n;1} & m_{22}M_{n;2} \\ m_{13}M_{n;3} & m_{14}M_{n;4} & m_{23}M_{n;3} & m_{24}M_{n;4} \\ \hline m_{31}M_{n;1} & m_{32}M_{n;2} & m_{41}M_{n;1} & m_{42}M_{n;2} \\ m_{33}M_{n;3} & m_{34}M_{n;4} & m_{43}M_{n;3} & m_{44}M_{n;4} \end{array} \right] = \begin{bmatrix} M_{n+1;1} & M_{n+1;2} \\ M_{n+1;3} & M_{n+1;4} \end{bmatrix},$$

where

$$\mathbb{M}_n = \hat{\otimes}\mathbb{M}_2^{n-1} = \begin{bmatrix} M_{n;1} & M_{n;2} \\ M_{n;3} & M_{n;4} \end{bmatrix}.$$

Here, the following convention is adopted,

$$\hat{\otimes}\mathbb{M}_2^0 = \mathbb{E}_{2\times 2}.$$

From Theorem 3.1, we can obtain results for $\mathbb{T}_{x;2\times m_2\times 2}$ as $\mathbb{T}_n$ in Theorem 3.1 in [5]. Indeed, we have

Theorem 3.8. Let $\mathbb{T}_{x;2\times 2\times 2}$ be a transition matrix given by (3.3.1) and (3.3.2). Then, for higher order transition matrices $\mathbb{T}_{x;2\times m_2\times 2}$, $m_2 \geq 3$, we have the following three equivalent expressions as follows:
(I) $\mathbb{T}_{x;2\times m_2\times 2}$ can be decomposed into m_2 successive 4×4 matrices

$$\mathbb{T}_{x;2\times m_2\times 2} \quad = \quad [T_{x;2\times m_2\times 2;j_1}]_{4\times 4},$$

where $1 \leq j_1 16$. For fixed $1 \leq j_1, j_2, \ldots, j_k \leq 16$,

$$T_{x;2\times m_2\times 2;j_1 j_2 \ldots j_k} = [T_{x;2\times m_2\times 2;j_1 j_2\ldots j_k j_{k+1}}]_{4\times 4},$$

where $1 \leq j_{k+1} 16$ and $1 \leq k \leq m_2 - 1$. For fixed $j_1, j_2, \ldots, j_{m_2-1} \in \{1, 2, \ldots, 16\}$,

$$T_{x;2\times m_2\times 2;j_1 j_2\ldots j_{m_2-1}} = [t_{y;2\times m_2\times 2;j_1 j_2\ldots j_{m_2}}]_{4\times 4},$$

where $t_{y;2\times m_2\times 2;j_1 j_2\ldots j_{m_2}}$ is defined in (3.3.3).
(II) Starting from

$$\mathbb{T}_{x;2\times 2\times 2} = [T_{x;2\times 2\times 2;j_1}]_{4\times 4}$$

and

$$T_{x;2\times 2\times 2;j_1} = [t_{y;2\times 2\times 2;j_1 j_2}]_{4\times 4},$$

for $m_2 \geq 3$,$\mathbb{T}_{x;2\times m_2\times 2}$ can be obtained from $\mathbb{T}_{x;2\times(m_2-1)\times 2}$ by replacing $T_{x;2\times 2\times 2;j_1}$ with

$$(T_{x;2\times 2\times 2;j_1})_{4\times 4} \circ (\mathbb{T}_{x;2\times 2\times 2})_{4\times 4}. \tag{3.3.10}$$

(III) For $m_2 \geq 3$,

$$\begin{aligned} \mathbb{T}_{x;2\times m_2\times 2} \quad = \quad & (\mathbb{T}_{x;2\times(m_2-1)\times 2})_{2^{2(m_2-1)}\times 2^{2(m_2-1)}} \\ & \circ (E_{2^{2(m_2-2)}} \otimes \mathbb{T}_{x;2\times 2\times 2})_{2^{2m_2}\times 2^{2m_2}}, \end{aligned} \tag{3.3.11}$$

where E_{2^k} is the $2^k \times 2^k$ matrix with 1 as its entries.

Proof. (I) The proof is to simply replace $A_{x;2\times m_2\times 2;j_1 j_2\ldots j_k}$ and $a_{x;2\times m_2\times 2;j_1 j_2\ldots j_k}$ by $T_{x;2\times m_2\times 2;j_1 j_2\ldots j_k}$ and $t_{x;2\times m_2\times 2;j_1 j_2\ldots j_k}$ in Theorem 3.1 respectively.
(II) follows from (I) directly.
(III) follows from (I), we have $\mathbb{T}_{x;2\times m_2\times 2} = [T_{x;2\times m_2\times 2;j_1}]$, $1 \leq j_1 \leq 2^4$. And by (I), we get following formula

$$\begin{aligned} \mathbb{T}_{x;2\times m_2\times 2} \quad & = \quad [a_{y;2\times 2\times 2;j_1 j_2} T_{x;2\times(m_2-1)\times 2;j_2}] \\ & = \quad (\mathbb{T}_{x;2\times(m_2-1)\times 2})_{2^{2(m_2-1)}\times 2^{2(m_2-1)}} \hat{\otimes} [E_{2^{2(m_2-2)}} \otimes \mathbb{T}_{x;2\times 2\times 2}]. \end{aligned}$$

The proof is complete. $\square$

Remark 3.9. As mentioned in Remark 3.2 we have the following formula but the detailed proof is omitted for brevity.

$$\begin{aligned}
T_{\hat{x};2\times 2\times m_3} &= [t_{z;2\times 2\times m_3;k_1k_2\ldots k_{m_3-1}k_{m_3}}]_{2^{m_3}\times 2^{m_3}}\\
T_{y;m_1\times 2\times 2} &= [t_{x;m_1\times 2\times 2;i_1i_2\ldots i_{m_1-1}i_{m_1}}]_{2^{m_1}\times 2^{m_1}}\\
T_{\hat{y};2\times 2\times m_3} &= [t_{\hat{z};2\times 2\times m_3;k_1k_2\ldots k_{m_3-1}k_{m_3}}]_{2^{m_3}\times 2^{m_3}}\\
T_{z;m_1\times 2\times 2} &= [t_{\hat{x};m_1\times 2\times 2;i_1i_2\ldots i_{m_1-1}i_{m_1}}]_{2^{m_1}\times 2^{m_1}}\\
T_{\hat{z};2\times m_2\times 2} &= [t_{\hat{y};2\times m_2\times 2;j_1j_2\ldots j_{m_2-1}j_{m_2}}]_{2^{m_2}\times 2^{m_2}}
\end{aligned}$$

Now, the transition matrix $\mathbb{T}_{\hat{x};2\times m_2\times 2}$, with respect to ordering matrix $\mathbb{A}_{\hat{x};2\times m_2\times 2}$. Additionally, by using Theorem 3.4, we have

Theorem 3.10.

$$\mathbb{T}_{\hat{x};2\times m_2\times 2} = \mathbb{P}^t_{x;2\times m_2\times 2}\mathbb{T}_{x;2\times m_2\times 2}\mathbb{P}_{x;2\times m_2\times 2}. \tag{3.3.12}$$

Proof. The proof is to simply replaced $a_{y;2\times m_2\times 2;j_1j_2\ldots j_{m_2}}$ by $t_{y;2\times m_2\times 2;j_1j_2\ldots j_{m_2}}$ in Theorem 3.4. □

By applying Theorem 3.5, transition matrix $\mathbb{T}_{\hat{x};2\times m_2\times m_3}$ can be obtained from $\mathbb{T}_{\hat{x};2\times m_2\times 2}$. According to (3.2.57), we obtained the transition matrix

$$\mathbb{T}_{\hat{x};2\times m_2\times 2} = [T_{\hat{x};2\times m_2\times 2;k_1}] \tag{3.3.13}$$

and

$$T_{\hat{x};2\times m_2\times 2;k_1} = [t_{z;2\times m_2\times 2;k_1k_2}]. \tag{3.3.14}$$

Therefore, we have

Theorem 3.11. Let $\mathbb{T}_{\hat{x};2\times m_2\times 2}$ be a transition matrix given by (3.3.13) and (3.3.14). Then, for higher order transition matrices $\mathbb{T}_{\hat{x};2\times m_2\times m_3}$, $m_2 \geq 3$, we have the following three equivalent expressions as follows:
(I) $\mathbb{T}_{\hat{x};2\times m_2\times m_3}$ can be decomposed into m_3 successive $2^{m_2}\times 2^{m_2}$ matrices:

$$\mathbb{T}_{\hat{x};2\times m_2\times m_3} = [T_{\hat{x};2\times m_2\times m_3;k_1}]_{2^{m_2}\times 2^{m_2}},$$

where $1 \leq k_1 \leq 2^{2m_2}$. For fixed $1 \leq k_1, k_2, \ldots, k_l \leq 2^{2m_2}$,

$$T_{\hat{x};2\times m_2\times m_3;k_1k_2\ldots k\ell} = [T_{\hat{x};2\times m_2\times m_3;k_1k_2\ldots k_\ell k_{\ell+1}}]_{2^{m_2}\times 2^{m_2}},$$

where $1 \leq k_{\ell+1} = 1 \leq 2^{2m_2}$ and $1 \leq l \leq m_3 - 2$,

$$T_{\hat{x};2\times m_2\times m_3;k_1k_2\ldots k_{m_3-1}} = [t_{z;2\times m_2\times m_3;k_1k_2\ldots k_{m_3}}]_{2^{m_2}\times 2^{m_2}},$$

where $1 \le k_{m_3} \le 2^{2m_2}$ and by (3.2.58)

$$t_{z;2\times m_2\times m_3;k_1k_2\ldots k_{m_3}} = \prod_{l=1}^{m_3-1} t_{z;2\times m_2\times 2;k_lk_{l+1}}. \tag{3.3.15}$$

(II) For any $m_3 \ge 3$, $\mathbb{T}_{\hat{x};2\times m_2\times m_3}$ can be obtained from $\mathbb{T}_{\hat{x};2\times m_2\times(m_3-1)}$ by replacing $T_{\hat{x};2\times m_2\times 2;k_1}$ with

$$(T_{\hat{x};2\times m_2\times 2;k_1})_{2^{m_2}\times 2^{m_2}} \circ (\mathbb{T}_{\hat{x};2\times m_2\times 2})_{2^{m_2}\times 2^{m_2}}. \tag{3.3.16}$$

(III) Furthermore, for $m_3 \ge 3$ we have

$$\begin{aligned}\mathbb{T}_{\hat{x};2\times m_2\times m_3} &= (\mathbb{T}_{\hat{x};2\times m_2\times(m_3-1)})_{2^{m_2(m_3-1)}\times 2^{m_2(m_3-1)}} \\ &\circ(E_{2^{m_2(m_3-2)}} \otimes \mathbb{T}_{\hat{x};2\times m_2\times 2})_{2^{m_2(m_3-1)}\times 2^{m_2(m_3-1)}}.\end{aligned} \tag{3.3.17}$$

The proof closely resembles that when proving Theorem 3.1 and Theorem 3.8. Details of the proof are omitted for brevity.

Remark 3.12. As mentioned in Remark 3.6, we also have the following formula but the detailed proof is omitted for brevity.

$$\begin{aligned}
T_{x;2\times m_2\times m_3} &= [t_{y;2\times m_2\times m_3;j_1j_2\ldots j_{m_2}}]_{2^{m_2m_3}\times 2^{m_2m_3}}\\
T_{\hat{y};m_1\times 2\times m_3} &= [t_{\hat{z};m_1\times 2\times m_3;k_1k_2\ldots k_{m_3}}]_{2^{m_1m_3}\times 2^{m_1m_3}}\\
T_{y;m_1\times 2\times m_3} &= [t_{x;m_1\times 2\times m_3;i_1i_2\ldots i_{m_1}}]_{2^{m_1m_3}\times 2^{m_1m_3}}\\
T_{\hat{z};m_1\times m_2\times 2} &= [t_{\hat{y};m_1\times m_2\times 2;j_1j_2\ldots j_{m_2}}]_{2^{m_1m_2}\times 2^{m_1m_2}}\\
T_{z;m_1\times m_2\times 2} &= [t_{\hat{x};m_1\times m_2\times 2;i_1i_2\ldots i_{m_1}}]_{2^{m_1m_2}\times 2^{m_1m_2}}
\end{aligned}$$

Finally, the spatial entropy $h(\mathcal{B})$ can be computed through the maximum eigenvalue $\lambda_{m,n}$ of $\mathbb{T}_{\hat{x};2\times m_2\times m_3}$. Indeed, we have

Theorem 3.13. Let $\lambda_{\hat{x};2,m_2,m_3}$ be the maximum eigenvalue of $\mathbb{T}_{\hat{x};2\times m_2\times m_3}$, then

$$h(\mathcal{B}) = \lim_{m_2,m_3\to\infty} \frac{\log \lambda_{\hat{x};2,m_2,m_3}}{m_2m_3}. \tag{3.3.18}$$

Proof. By the same arguements as in [16], the limit (3.1.1) is well-defined and exists. From the construction of $\mathbb{T}_{\hat{x};2\times m_2\times m_3}$, we observe that for $m_2 \ge 2$ and $m_3 \ge 2$,

$$\begin{aligned}\Gamma_{\hat{x};m_1\times m_2\times m_3}(\mathcal{B}) &= \sum_{1\le i,j\le 2^{m_2m_3}} (\mathbb{T}^{m_1-1}_{\hat{x};2\times m_2\times m_3})_{ij}\\ &= \sharp(\mathbb{T}^{m_1-1}_{\hat{x};2\times m_2\times m_3}).\end{aligned}$$

As in one dimensional case, we have

$$\lim_{m_1\to\infty}\frac{\log\sharp(\mathbb{T}^{m_1-1}_{\hat{x};2\times m_2\times m_3})}{m_1}=\log\lambda_{\hat{x};2,m_2,m_3},$$

e.g., [4]. Therefore,

$$\begin{aligned}h(\mathcal{B}) &= \lim_{m_1,m_2,m_3\to\infty}\frac{\log\Gamma_{\hat{x};m_1\times m_2\times m_3}(\mathcal{B})}{m_1m_2m_3}\\ &= \lim_{m_2,m_3\to\infty}\frac{1}{m_2m_3}(\lim_{m_1\to\infty}\frac{\log\Gamma_{\hat{x};m_1\times m_2\times m_3}(\mathcal{B})}{m_1})\\ &= \lim_{m_2,m_3\to\infty}\frac{\log\lambda_{\hat{x};2,m_2,m_3}}{m_2m_3}\end{aligned}$$

The proof is complete. □

Remark 3.14. Let $\lambda_{x;2,m_2,m_3}$, $\lambda_{\hat{y};m_1,2,m_3}$, $\lambda_{y;m_1,2,m_3}$, $\lambda_{\hat{z};m_1,m_2,2}$ and $\lambda_{z;m_1,m_2,2}$ be the maximum eigenvalue of $\mathbb{T}_{x;2\times m_2\times m_3}$, $\mathbb{T}_{\hat{y};m_1\times 2\times m_3}$, $\mathbb{T}_{y;m_1\times 2\times m_3}$, $\mathbb{T}_{\hat{z};m_1\times m_2\times 2}$ and $\mathbb{T}_{z;m_1\times m_2\times 2}$ respectively, then it can be shown that

$$\begin{aligned}h(\mathcal{B}) &= \lim_{m_2,m_3\to\infty}\frac{\log\lambda_{x;2,m_2,m_3}}{m_2m_3}\\ &= \lim_{m_1,m_3\to\infty}\frac{\log\lambda_{\hat{y};m_1,2,m_3}}{m_1m_3}\\ &= \lim_{m_1,m_3\to\infty}\frac{\log\lambda_{y;m_1,2,m_3}}{m_1m_3}\\ &= \lim_{m_1,m_2\to\infty}\frac{\log\lambda_{\hat{z};m_1,m_2,2}}{m_1m_2}\\ &= \lim_{m_1,m_2\to\infty}\frac{\log\lambda_{z;m_1,m_2,2}}{m_1m_2}.\end{aligned}$$

but the detailed proof is omitted here for brevity.

3.3.2 Computation of $\lambda_{m,n}$ and entropies

From the last subsection, we obtained a systematic means of writing down $\mathbb{T}_{\hat{x};2\times m_2\times m_3}$ from $\mathbb{T}_{x;2\times 2\times 2}$. As in a two dimensional case [4], a recursion formulas for $\lambda_{\hat{x};2,m_2,m_3}$ can be obtained in special structure. To demonstrate the methods developed in the last subsection, we provide an illustrative example in which $\mathbb{T}_{\hat{x};2\times m_2\times m_3}$ and $\lambda_{\hat{x};2,m_2,m_3}$ can be derived explicitly. More complete results will be appeared later.

Denoted by

$$G = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \textit{ and } E = E_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \tag{3.3.19}$$

and let

$$\begin{aligned} \mathbb{T}_{x;2\times 2\times 2} &= \otimes(G \otimes E)^2, \\ &= (G \otimes E) \otimes (G \otimes E). \end{aligned} \tag{3.3.20}$$

Proposition 3.15. Let $\mathbb{T}_{x;2\times 2\times 2}$ be in (3.3.19) and (3.3.20). Then,

$$\begin{array}{ll} (i) & \mathbb{T}_{x;2\times m_2\times 2} = \otimes(G \otimes E)^{m_2}, \\ (ii) & \mathbb{T}_{\hat{x};2\times m_2\times 2} = (\otimes G)^{m_2} \otimes (\otimes E)^{m_2}, \\ (iii) & \mathbb{T}_{\hat{x};2\times m_2\times m_3} = (\otimes G)^{m_2(m_3-1)} \otimes (\otimes E)^{m_2}. \end{array} \tag{3.3.21}$$

Furthermore, for the maximum eigenvalue $\lambda_{\hat{x};2,m_2,m_3}$ of $\mathbb{T}_{\hat{x};2\times m_2\times m_3}$, we have the following recursion formulas:

$$\lambda_{\hat{x};2,m_2+1,m_3} = 2g^{m_3-1}\lambda_{\hat{x};2,m_2,m_3} \tag{3.3.22}$$

and

$$\lambda_{\hat{x};2,m_2,m_3+1} = g^{m_2}\lambda_{\hat{x};2,m_2,m_3} \tag{3.3.23}$$

for $m_2, m_3 \geq 2$ with

$$\lambda_{\hat{x};2,2,2} = (2g)^2. \tag{3.3.24}$$

The spatial entropy is

$$h(\mathbb{T}_{x;2\times 2\times 2}) = \log g, \tag{3.3.25}$$

where $g = \frac{1+\sqrt{5}}{2}$, the golden-mean.

Proof. The proof is only described briefly, and the details are omitted for brevity.
(i) can be proved by Theorem 3.8 and induction on m. Indeed, by (3.3.11), we have

$$\begin{aligned} \mathbb{T}_{x;2\times 3\times 2} &= (\mathbb{T}_{x;2\times 2\times 2})_{4\times 4} \circ (E_{2^2} \otimes \mathbb{T}_{x;2\times 2\times 2})_{4\times 4} \\ &= (G \otimes E \otimes G \otimes E)_{4\times 4} \circ (E \otimes E \otimes (G \otimes E \otimes G \otimes E))_{4\times 4} \\ &= (G \circ E) \otimes (E \circ E) \otimes (G \circ G) \otimes (E_{2\times 2} \circ (E \otimes G \otimes E))_{2\times 2} \\ &= \otimes(G \otimes E)^3. \end{aligned}$$

Assume that $\mathbb{T}_{x;2\times(m_2-1)\times 2} = \otimes(G\otimes E)^{m_2-1}$. Then by (3.3.11) again, we have

$$\begin{aligned}
&\mathbb{T}_{x;2\times m_2\times 2} = (\mathbb{T}_{x;2\times(m_2-1)\times 2}) \circ (\otimes(E)^{2(m_2-2)} \otimes \mathbb{T}_{x;2\times 2\times 2}))\\
&= (\otimes(G\otimes E)^{m_2-1})_{2^{2m_2-2}\times 2^{2m_2-2}} \circ \ ((\otimes(E)^{m_2-2}) \otimes (\otimes(G\otimes E)^2))_{2^{2m_2-2}\times 2^{2m_2-2}}\\
&= (\otimes(G\otimes E)^{m_2-2} \otimes (G\otimes E))_{2^{2m_2-2}\times 2^{2m_2-2}}\\
&\quad \circ (\otimes(E\otimes E)^{m_2-2} \otimes (G\otimes E)\otimes(G\otimes E))_{2^{2m_2-2}\times 2^{2m_2-2}}\\
&= \otimes[(G\circ E)\otimes(E\circ E)]^{m_2-2} \otimes (G\circ G)\otimes(E\circ(E\otimes G\otimes E))\\
&= \otimes(G\otimes E)^{m_2-2}\otimes(G\otimes E)\otimes(G\otimes E)\\
&= \otimes(G\otimes E)^{m_2}.
\end{aligned}$$

(ii) The following property for matrices is needed and the detailed proof omitted: For any two 2×2 matrices A and B, we have

$$P(A\otimes B)P = B\otimes A, \tag{3.3.26}$$

where P is given in (3.2.51). We also prove in (3.3.21) by induction on m_2. When $m_2 = 2$, by Theorem 3.10,

$$\begin{aligned}
\mathbb{T}_{\hat{x};2\times 2\times 2} &= \mathbb{P}^t_{x;2\times 2\times 2}\mathbb{T}_{x;2\times 2\times 2}\mathbb{P}_{x;2\times 2\times 2}\\
&= (P_{4;2})^t\mathbb{T}_{x;2\times 2\times 2}P_{4;2}\\
&= (I_2\otimes P\otimes I_2)((G\otimes E)\otimes(G\otimes E))(I_2\otimes P\otimes I_2)\\
&= G\otimes(P(E\otimes G)P)\otimes E\\
&= G\otimes G\otimes E\otimes E
\end{aligned}$$

by (3.3.26).

Now, assume that (3.3.21) holds for m_2-1, i.e.

$$\mathbb{T}_{\hat{x};2\times(m_2-1)\times 2} = (\otimes(G)^{m_2-1})\otimes(\otimes(E)^{m_2-1}).$$

Then

$$\begin{aligned}
&\mathbb{T}_{\hat{x};2\times m_2\times 2}\\
&= \mathbb{P}^t_{x;2\times m_2\times 2}\mathbb{T}_{x;2\times m_2\times 2}\mathbb{P}_{x;2\times m_2\times 2}\\
&= [(P_{2m_2;2}P_{2m_2;4}\cdots P_{2m_2;2m_2-2})(P_{2m_2;3}P_{2m_2;5}\cdots P_{2m_2;2m_2-3})\cdots(P_{2m_2;m})]^t\\
&\quad \mathbb{T}_{x;2\times m_2\times 2}[(P_{2m_2;2}P_{2m_2;4}\cdots P_{2m_2;2m_2-2})(P_{2m_2;3}P_{2m_2;5}\cdots P_{2m_2;2m_2-3})\cdots(P_{2m_2;m})]\\
&= (P_{2m_2;m})\cdots(P_{2m_2;3}P_{2m_2;5}\cdots P_{2m_2;2m_2-3})[(P_{2m_2;2}P_{2m_2;4}\cdots P_{2m_2;2m_2-2})\\
&\quad \otimes(G\otimes E)^{m_2})(P_{2m_2;2}P_{2m_2;4}\cdots P_{2m_2;2m_2-2})](P_{2m;3}P_{2m;5}\cdots P_{2m;2m_2-3})\cdots(P_{2m;m})\\
&= (P_{2m;m})\cdots(P_{2m;3}P_{2m;5}\cdots P_{2m;2m_2-3})[G\otimes(\otimes(G\otimes E)^{m_2-1})\otimes E]\\
&\quad (P_{2m;3}P_{2m;5}\cdots P_{2m;2m_2-3})\cdots(P_{2m;m})\\
&= G\otimes\{(P_{2(m_2-1);m_2-1})\cdots(P_{2(m_2-1);2}P_{2(m_2-1);4}\cdots P_{2(m_2-1);2(m_2-1)-2})[\otimes(G\otimes E)^{m_2-1}]\\
&\quad (P_{2(m_2-1);2}P_{2(m_2-1);4}\cdots P_{2(m_2-1);2(m_2-1)-2})\cdots(P_{2m_2-1;m_2-1})\}\otimes E\\
&= G\otimes(\mathbb{P}^t_{x;2\times(m_2-1)\times 2}\mathbb{T}_{x;2\times(m_2-1)\times 2}\mathbb{P}_{x;2\times(m_2-1)\times 2})\otimes E\\
&= G\otimes\mathbb{T}_{\hat{x};2\times(m_2-1)\times 2}\otimes E\\
&= G\otimes((\otimes(G)^{m_2-1})\otimes(\otimes(E)^{m_2-1}))\otimes E\\
&= (\otimes(G)^{m_2})\otimes(\otimes(E)^{m_2}).
\end{aligned}$$

(iii) For a fixed m_2, we prove the results by induction on $m_3 \geq 2$. Assume that (3.3.21) holds for $m_3 - 1$, i.e.,

$$\mathbb{T}_{\hat{x};2\times m_2\times(m_3-1)} = (\otimes(G)^{m_2\times(m_3-2)}) \otimes (\otimes(E)^{m_2}).$$

Then, by (3.3.17), we have

$$\begin{aligned}
\mathbb{T}_{\hat{x};2\times m_2\times m_3} &= \mathbb{T}_{\hat{x};2\times m_2\times(m_3-1)} \circ ((\otimes(E)^{m_2(m_3-2)}) \otimes \mathbb{T}_{\hat{x};2\times m_2\times 2}) \\
&= ((\otimes(G)^{m_2(m_3-2)}) \otimes (\otimes(E)^{m_2})) \circ ((\otimes(E)^{m_2(m_3-2)}) \otimes (\otimes(G)^{m_2}) \otimes (\otimes(E)^{m_2})) \\
&= (\otimes(G)^{m_2(m_3-2)}) \otimes (\otimes(G)^{m_2}) \otimes (\otimes(E)^{m_2}) \\
&= (\otimes(G)^{m_2(m_3-1)}) \otimes (\otimes(E)^{m_2}).
\end{aligned}$$

As for maximum eigenvalue $\lambda_{\hat{x};2,m_2,m_3}$, verifying (3.3.24) is easy. To show (3.3.22) for fixed m_3, by using (3.3.21), we have

$$\begin{aligned}
\mathbb{T}_{\hat{x};2\times(m_2+1)\times m_3} &= (\otimes(G)^{(m_2+1)(m_3-1)}) \otimes (\otimes(E)^{m_2+1}) \\
&= (\otimes(G)^{m_3-1}) \otimes (\otimes(G)^{m_2(m_3-1)}) \otimes (\otimes(E)^{m_2}) \otimes E \\
&= (\otimes(G)^{m_3-1}) \otimes \mathbb{T}_{\hat{x};2\times m_2\times m_3} \otimes E,
\end{aligned}$$

which implies

$$\lambda_{\hat{x};2,m_2+1,m_3} = 2g^{n-1}\lambda_{\hat{x};2,m_2,m_3},$$

see [12].
Similarly, for a fixed m_2, to prove (3.3.23), by using (3.3.21) again, we have

$$\begin{aligned}
\mathbb{T}_{\hat{x};2\times m_2\times(m_3+1)} &= (\otimes(G)^{m_2 m_3}) \otimes (\otimes(E)^{m_2}) \\
&= (\otimes(G)^{m_2}) \otimes (\otimes(G)^{m_2(m_3-1)}) \otimes (\otimes(E)^{m_2}) \\
&= (\otimes(G)^{m_2}) \otimes \mathbb{T}_{\hat{x};2\times m_2\times m_3},
\end{aligned}$$

which implies

$$\lambda_{\hat{x};2,m_2,m_3+1} = g^{m_2}\lambda_{\hat{x};2,m_2,m_3}.$$

Finally, (3.3.25) follows from (3.3.22), (3.3.23) and Theorem 3.13. The proof is thus complete. □

§ 3.4 Connecting Operator

As stated in the introduction, in this section we will introduce the connecting operator and to use it to derive a recursive formula between an elementary pattern of order (m, n). And use it to yield a lower bound on entropy.

3.4.1 Connecting Operator in z-direction

This subsection derives connecting operators and investigates their properties. For brevity, we just discuss the connecting operator in z-direction and the other cases are similar and we will state them in Remarks follows. For clarity, such as in the former section two symbols on lattice $\mathbb{Z}_{2\times 2\times 2}$ are examined first.

As state in Theorem 3.5, the ordering matrix $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ can be represented by $A_{\hat{x};2\times m_2\times m_3;\alpha}$, where $1 \leq \alpha \leq 2^{2m_2}$, is a $2^{m_2(m_3-1)} \times 2^{m_2(m_3-1)}$ matrix.

For matrices multiplication, the indices of $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ are conveniently expressed as

$$\begin{bmatrix} A_{\hat{x};2\times m_2\times m_3;11} & A_{\hat{x};2\times m_2\times m_3;12} & \cdots & A_{\hat{x};2\times m_2\times m_3;12^{m_2}} \\ A_{\hat{x};2\times m_2\times m_3;21} & A_{\hat{x};2\times m_2\times m_3;22} & \cdots & A_{\hat{x};2\times m_2\times m_3;22^{m_2}} \\ \vdots & \vdots & \ddots & \vdots \\ A_{\hat{x};2\times m_2\times m_3;2^{m_2}1} & A_{\hat{x};2\times m_2\times m_3;2^{m_2}2} & \cdots & A_{\hat{x};2\times m_2\times m_3;2^{m_2}2^{m_2}}. \end{bmatrix} \tag{3.4.1}$$

Clearly, $A_{\hat{x};2\times m_2\times m_3;\alpha} = A_{\hat{x};2\times m_2\times m_3;\beta_1\beta_2}$, where $\alpha = \alpha(\beta_1,\beta_2) = 2^{m_2}(\beta_1 - 1) + \beta_2$. For $m_1 \geq 2$, the elementary pattern in the entries of $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times m_3}$ is represented by $A_{\hat{x};2\times 2\times 2;\beta_1\beta_2}A_{\hat{x};2\times 2\times 2;\beta_2\beta_3}\cdots A_{\hat{x};2\times 2\times 2;\beta_{m_1}\beta_{m_1+1}}$ where $\beta_r \in \{1,2,\cdots,2^{m_2}\}$. A lexicographic order for multiple indices $I_{m_1+1} = (\beta_1\beta_2\cdots\beta_m\beta_{m_1+1})$ is introducing, using

$$\mathcal{K}(I_{m_1+1}) = 1 + \sum_{r=2}^{m_1} 2^{m_2(m_1-r)}(\beta_r - 1). \tag{3.4.2}$$

Now, $A^{(k)}_{\hat{x};m_1+1,m_2,m_3;\alpha}$ could be represented by

$$A_{\hat{x};2\times m_2\times m_3;\beta_1\beta_2}A_{\hat{x};2\times m_2\times m_3;\beta_2\beta_3}\cdots A_{\hat{x};2\times m_2\times m_3;\beta_{m_1}\beta_{m_1+1}}, \tag{3.4.3}$$

where

$$\alpha = \alpha(\beta_1,(\beta_{m_1}+1))) = 2^{m_2}(\beta_1 - 1) + \beta_{m_1+1} \tag{3.4.4}$$

and

$$k = \mathcal{K}(I_{m_1+1}) \tag{3.4.5}$$

is given in (3.4.2). Therefore, $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times m_3}$ can be expressed as

$$[A_{\hat{x};m_1,m_2,m_3;\alpha}]_{2^{m_2}\times 2^{m_2}}, \tag{3.4.6}$$

where $1 \leq \alpha \leq 2^{2m_2}$ and

$$A_{\hat{x};m_1,m_2,m_3;\alpha} = \sum_{k=1}^{2^{m_2(m_1-1)}} A^{(k)}_{\hat{x};m_1,m_2,m_3;\alpha}. \tag{3.4.7}$$

Furthermore,

$$X_{\hat{x};m_1,m_2,m_3;\alpha} = (A^{(k)}_{\hat{x};m_1,m_2,m_3;\alpha})^t, \tag{3.4.8}$$

where $1 \leq k \leq 2^{m_2(m_1-1)}$, $X_{\hat{x};m_1,m_2,m_3;\alpha}$ is a $2^{m_2(m_1-1)}$ column vector that consists of all elementary patterns in $A_{\hat{x};m_1,m_2,m_3;\alpha}$. The ordering matrix $\mathbb{X}_{\hat{x};m_1,m_2,m_3}$ of $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times m_3}$ is now defined by

$$[X_{\hat{x};m_1,m_2,m_3;\alpha}]_{2^{m_2}\times 2^{m_2}}, \tag{3.4.9}$$

where $1 \leq \alpha \leq 2^{2m_2}$. The ordering matrix $\mathbb{X}_{\hat{x};m_1,m_2,m_3}$ allows the elementary patterns to be tracked during the reduction from $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times(m_3+1)}$ to $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times m_3}$. This careful book-keeping provides a systematic way to generate the admissible patterns and in Section 3.4.2, lower-bound estimates of spatial entropy.

The following simplest example is studied first to illustrate the above concept.

Example 3.16. For $m_1 = 2$, $m_2 = 3$, $m_3 = 3$, the following can be easily be verified;

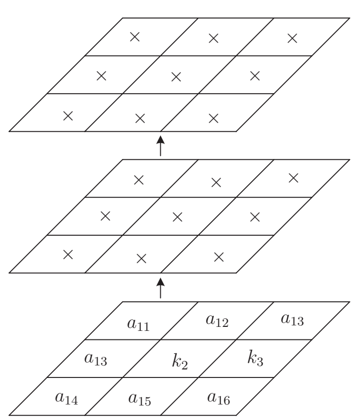
$$\mathbb{A}^2_{\hat{x};2\times 3\times 3} = [A_{\hat{x};2,3,3;\alpha_1}]_{2^3\times 2^3}, \tag{3.4.10}$$

where $1 \leq \alpha_1 \leq 2^6$ and

$$A_{\hat{x};2,3,3;\alpha_1} = \sum_{k=1}^{2^3} A^{(k)}_{\hat{x};2,3,3;\alpha_1}, \tag{3.4.11}$$

and for fixed α_1 and k the represented pattern of $A^{(k)}_{\hat{x};2,3,3;\alpha_1}$ are as the following form

(3.4.12)



.

If we defined the red symbol is equal to 1, white symbol is equal to 0, then $\alpha_1 = 2^5\alpha_{11} + 2^4\alpha_{12} + 2^3\alpha_{13} + 2^2\alpha_{14} + 2\alpha 15 + \alpha_{16} + 1$ and $k = 2^2k_1 + 2k_2 + k_3 + 1$. Therefore

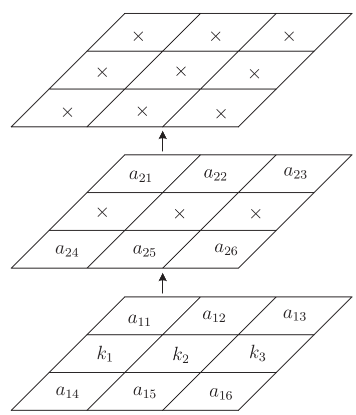
$$X_{\hat{x};2,3,3;\alpha_1} = (A^{(k)}_{\hat{x};2,3,3;\alpha_1})^t, \tag{3.4.13}$$

where $1 \leq k \leq 2^3$ and $1 \leq \alpha_1 \leq 2^6$. Define

$$X_{\hat{x};2,3,3;\alpha_1;\alpha_2} = (A^{(k)}_{\hat{x};2,3,3;\alpha_1;\alpha_2})^t, \tag{3.4.14}$$

where $1 \leq k \leq 2^3$ and $1 \leq \alpha_1, \alpha_2 \leq 2^6$ and the represented pattern is

(3.4.15) $$A^{(k)}_{\hat{x};2,3,3;\alpha_1;\alpha_2} =$$



.

Hence we get, for example

$$X_{\hat{x};2,3,3;1;1} = S_{\hat{x};m_3;2,3;11} \cdot X_{\hat{x};2,3,2;1}, \tag{3.4.16}$$

and the represented patterns of $S_{\hat{x};m_3;2,3;11}$

(3.4.17) $\quad {}_{2^3\times 2^3}.$

The above derivation indicates that $X_{\hat{x};2,3,3;\alpha_1;\alpha_2}$ can reduced to $X_{\hat{x};2,3,3;\alpha_2}$ via multiplication with connecting operator $S_{\hat{x};m_3;2,3;\alpha_1\alpha_2}$. This procedure can be extended to introduce the connecting operator $\mathbb{S}_{\hat{x};m_3;m_1m_2} = [S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}]$, where $1 \leq \alpha_1, \alpha_2 \leq 2^{2m_2}$, for all $m_1 \geq 2$, $m_2 \geq 2$.

Definition 3.17. For $m_1 \geq 2$, $m_2 \geq 2$, define

$$(\mathbb{C}_{\hat{x};m_3;m_1m_2})_{2^{2m_2}\times 2^{2m_2}} = (\mathbb{S}^{(r)}_{\hat{x};m_3;m_1m_2})_{2^{2m_2}\times 2^{2m_2}}, \tag{3.4.18}$$

where the row matrix $\mathbb{S}^{(r)}_{\hat{x};m_3;m_1m_2}$ of $\mathbb{S}_{\hat{x};m_3;m_1m_2}$ is defined in (3.2.23) and (3.2.24). And

$$\begin{aligned} &C_{\hat{x};m_3;m_1m_2;i_1i_2} \\ = \quad & [(\mathbb{T}_{z;2\times m_2\times 2;i_1})_{2^{m_2}\times 2^{m_2}} \circ (\mathbb{T}_{z;(m_1-1)\times m_2\times 2})_{2^{m_2}\times 2^{m_2}}]_{2^{(m_1-1)m_2}\times 2^{(m_1-1)m_2}} \\ \circ \quad & (E_{2^{(m_1-2)m_2}} \otimes ((\mathbb{T}^{(r)}_{z;2\times m_2\times 2})^{(c)}_{;i_2})_{2^{m_2}\times 2^{m_2}})_{2^{(m_1-1)m_2}\times 2^{(m_1-1)m_2}} \end{aligned} \tag{3.4.19}$$

where $(\mathbb{T}^{(r)}_{z;2\times m_2\times 2})^{(c)}_{;\alpha_2}$ is the α_2-th block of the matrix $(\mathbb{T}^{(r)}_{z;2\times m_2\times 2})^{(c)}$, $(\mathbb{T}^{(r)}_{z;2\times m_2\times 2})^{(c)}$ is the column matrix of $\mathbb{T}^{(r)}_{z;2\times m_2\times 2}$ and $\mathbb{T}^{(r)}_{z;2\times m_2\times 2}$ is the row matrix of $\mathbb{T}_{z;2\times m_2\times 2}$.

Remark 3.18. With the similar method, we also can defined the following connecting operators.

$$\begin{aligned}
& C_{x;m_2;m_1m_3;i_1i_2} \\
= \ & [(\mathbb{T}_{y;2\times 2\times m_3;i_1})_{2^{m_3}\times 2^{m_3}} \circ (\mathbb{T}_{z;(m_1-1)\times m_2\times 2})_{2^{m_3}\times 2^{m_3}}]_{2^{(m_1-1)m_3}\times 2^{(m_1-1)m_3}} \\
\circ \ & (E_{2^{(m_1-2)m_3}} \otimes ((\mathbb{T}^{(r)}_{y;2\times 2\times m_3})^{(c)}_{;i_2})_{2^{m_3}\times 2^{m_3}})_{2^{(m_1-1)m_3}\times 2^{(m_1-1)m_3}}
\end{aligned}$$

$$\begin{aligned}
& C_{\hat{y};m_3;m_1m_2;i_1i_2} \\
= \ & [(\mathbb{T}_{\hat{z};m_1\times 2\times 2;i_1})_{2^{m_1}\times 2^{m_1}} \circ (\mathbb{T}_{\hat{z};m_1\times (m_2-1)\times 2})_{2^{m_1}\times 2^{m_1}}]_{2^{(m_2-1)m_1}\times 2^{(m_2-1)m_1}} \\
\circ \ & (E_{2^{(m_2-2)m_1}} \otimes ((\mathbb{T}^{(r)}_{\hat{z};2\times (m_2-1)\times 2})^{(c)}_{;i_2})_{2^{m_1}\times 2^{m_1}})_{2^{(m_2-1)m_1}\times 2^{(m_2-1)m_1}}
\end{aligned}$$

$$\begin{aligned}
& C_{y;m_1;m_2m_3;i_1i_2} \\
= \ & [(\mathbb{T}_{x;2\times 2\times m_3;i_1})_{2^{m_3}\times 2^{m_3}} \circ (\mathbb{T}_{x;2\times (m_2-1)\times 2})_{2^{m_3}\times 2^{m_3}}]_{2^{(m_2-1)m_3}\times 2^{(m_2-1)m_3}} \\
\circ \ & (E_{2^{(m_2-2)m_3}} \otimes ((\mathbb{T}^{(r)}_{x;2\times 2\times m_3})^{(c)}_{;i_2})_{2^{m_3}\times 2^{m_3}})_{2^{(m_2-1)m_3}\times 2^{(m_2-1)m_3}}
\end{aligned}$$

$$\begin{aligned}
& C_{\hat{z};m_2;m_1m_3;i_1i_2} \\
= \ & [(\mathbb{T}_{\hat{y};m_1\times 2\times 2;i_1})_{2^{m_1}\times 2^{m_1}} \circ (\mathbb{T}_{\hat{y};m_1\times (m_3-1)\times 2})_{2^{m_1}\times 2^{m_1}}]_{2^{(m_3-1)m_1}\times 2^{(m_3-1)m_1}} \\
\circ \ & (E_{2^{(m_3-2)m_1}} \otimes ((\mathbb{T}^{(r)}_{\hat{y};2\times (m_3-1)\times 2})^{(c)}_{;i_2})_{2^{m_1}\times 2^{m_1}})_{2^{(m_3-1)m_1}\times 2^{(m_3-1)m_1}}
\end{aligned}$$

$$\begin{aligned}
& C_{z;m_1;m_2m_3;i_1i_2} \\
= \ & [(\mathbb{T}_{\hat{x};2\times m_2\times 2};i_1)_{2^{m_2}\times 2^{m_2}} \circ (\mathbb{T}_{\hat{x};2\times m_2\times (m_3-1)})_{2^{m_2}\times 2^{m_2}}]_{2^{(m_3-1)m_2}\times 2^{(m_3-1)m_2}} \\
\circ \ & (E_{2^{(m_3-2)m_2}} \otimes ((\mathbb{T}^{(r)}_{\hat{x};2\times m_2\times 2})^{(c)}_{;i_2})_{2^{m_2}\times 2^{m_2}})_{2^{(m_3-1)m_2}\times 2^{(m_3-1)m_2}}
\end{aligned}$$

Theorem 3.19. For any $m_2 \geq 2$, $m_3 \geq 2$ and $1 \leq i_1, i_2 \leq 2^{2m_2}$,

$$C_{\hat{x};m_3;m_1+1,m_2;i_1i_2} = [t_{\hat{x};2\times m_2\times 2;i_1i}C_{\hat{x};m_3;m_1m_2;ii_2}]_{2^{m_2}\times 2^{m_2}}, \tag{3.4.20}$$

where $q \leq i \leq 2^{2m_2}$

Proof. By Remark 3.12 and Theorem 3.11,

$$\mathbb{T}_{z;(m_1-1)\times m_2\times 2} = [T_{z;2\times m_2\times 2;i_1} \circ \mathbb{T}_{z;(m_1-2)\times m_2\times 2}],$$

where $1 \leq i_1 \leq 2^{2m_2}$. Therefore, by

$$\begin{aligned}
& C_{\hat{x};m_3;(m_1+1)m_2;i_1i_2} \\
= \ & [(T_{z;2\times m_2\times 2}) \circ \mathbb{T}_{z;(m_1-1)\times m_2\times 2}] \circ [E_{2^{(m_1-3)m_2}} \otimes (T^{(r)}_{z;2\times m_2\times 2})^{(c)}_{;i_2}] \\
= \ & [t_{\hat{x};2\times m_2\times 2;i_1i}(T_{z;2\times m_2\times 2;i} \circ \mathbb{T}_{z;(m_1-2)\times m_2\times 2})]_{2^{m_2}\times 2^{m_2}} \\
& \circ[E_{2^{m_2}} \otimes (E_{2^{(m_1-4)m_2}} \otimes (T^{(r)}_{z;2\times m_2\times 2})^{(c)}_{;i_2})] \\
= \ & [t_{\hat{x};2\times m_2\times 2;i_1i}C_{\hat{x};m_3;m_1m_2;i_1i_2}]_{2^{m_2}\times 2^{m_2}}
\end{aligned}$$

where $1 \leq i \leq 2^{2m_2}$. The proof is complete. $\square$

Notably, (3.4.20) implies $C_{\hat{x};m_3;m_1m_2;ij}$ is

$$t_{\hat{x};2\times m_2\times 2;i_1i_2}t_{\hat{x};2\times m_2\times 2;i_2i_3}\cdots t_{\hat{x};2\times m_2\times 2;i_{m_1}i_{m_1+1}}$$

with $i_1 = i$ and $i_{m+1} = j$. $C_{\hat{x};m_3;m_1m_2;ij}$ consist of all paths of length $m_1 + 1$ starting from i and ending at j. Indeed, the entries of $\mathbb{C}_{\hat{x};m_3;m_1m_2}$ and $\mathbb{T}_{z;(m_1+1)\times m_2\times 2}$ are the same. However, the arrangements are different.

In (3.4.3) substituting m_3 for m_3+1 and using (3.3.17), $A^{(k)}_{\hat{x};m_1,m_2,m_3+1;\alpha}$ could be represented by

$$\begin{aligned}&A_{\hat{x};2\times m_2\times(m_3+1);\beta_1\beta_2}A_{\hat{x};2\times m_2\times(m_3+1);\beta_2\beta_3}\cdots A_{\hat{x};2\times m_2\times(m_3+1);\beta_{m_1}\beta_{m_1+1}}\\ &\prod_{j=1}^{m_1}[a_{\hat{x};2\times m_2\times 2;\alpha_j\hat{\alpha}}A_{\hat{x};2\times m_2\times m_3;\hat{\beta}_1\hat{\beta}_2}]_{2^{m_2}\times 2^{m_2}}\,,\end{aligned} \tag{3.4.21}$$

where $1 \le \hat{\beta}_1, \hat{\beta}_2 \le 2^{m_2}$ and $\alpha_j = \alpha([\beta_j, \beta_{j+1}])$ and $\hat{\alpha} = \alpha(\hat{\beta}_1, \hat{\beta}_2)$ for $1 \le j \le m_1$.

After m_1 matrix multiplications are executed in (3.4.21),

$$A^{(k)}_{\hat{x};m_1,m_2,m_3+1;\alpha_1} = [A^{(k)}_{\hat{x};m_1,m_2,m_3+1;\alpha_1;\alpha_2}]_{2^{m_2}\times 2^{m_2}}\,, \tag{3.4.22}$$

where $1 \le \alpha_2 \le 2^{2m_2}$ and $A^{(k)}_{\hat{x};m_1,m_2,m_3+1;\alpha_1;\alpha_2}$ could be represented by

$$\sum_{l=1}^{2^{m_2(m_1-1)}} K(\hat{x}, m_1m_2; \alpha_1\alpha_2; k, l)A^{(l)}_{\hat{x};m_1,m_2,m_3;\alpha_2} \tag{3.4.23}$$

which is a linear combination of $A^{(l)}_{\hat{x};m_1,m_2,m_3;\alpha_2}$ with the coefficients $K(\hat{x}, m_1m_2; \alpha_1\alpha_2; k, l)$ which are products of $a_{\hat{x};2\times m_2\times 2;\alpha_j\hat{\alpha}}$, $1 \le j \le m_1$. $K(\hat{x}, m_1m_2; \alpha_1\alpha_2; k, l)$ must be studied in more details. Note that

$$A^{m_1}_{\hat{x};2\times m_2\times(m_3+1)} = [A_{\hat{x};m_1,m_2,m_3+1;\alpha_1}]_{2^{m_2}\times 2^{m_2}} \tag{3.4.24}$$

where $1 \le \alpha_1 \le 2^{2m_2}$,

$$A_{\hat{x};m_1,m_2,m_3+1;\alpha_1} = \sum_{k=1}^{2^{m_2(m_1-1)}} A^{(k)}_{\hat{x};m_1,m_2,m_3+1;\alpha_1} \tag{3.4.25}$$

and

$$\sum_{k=1}^{2^{m_2(m_1-1)}} A^{(k)}_{\hat{x};m_1,m_2,(m_3+1);\alpha_1} = [\sum_{k=1}^{2^{m_2(m_1-1)}} A^{(k)}_{\hat{x};m_1,m_2,(m_3+1);\alpha_1;\alpha_2}]_{2^{m_2}\times 2^{m_2}}\,, \tag{3.4.26}$$

where $1 \leq \alpha_2 \leq 2^{2m_2}$. Now, $X_{\hat{x};m_1,m_2,m_3+1;\alpha_1;\alpha_2}$ is defined as

$$X_{\hat{x};m_1+1,m_2,m_3+1;\alpha_1;\alpha_2} = (A^{(k)}_{\hat{x};m_1+1,m_2,m_3+1;\alpha_1;\alpha_2})^t. \tag{3.4.27}$$

And from (3.4.23) and (3.4.27),

$$X_{\hat{x};m_1,m_2,m_3+1;\alpha_1;\alpha_2} = \mathbb{K}(\hat{x}, m_1m_2; \alpha_1\alpha_2)X_{\hat{x};m_1,m_2,m_3+1;\alpha_2} \tag{3.4.28}$$

where

$$\mathbb{K}(\hat{x}, m_1m_2; \alpha_1\alpha_2) = (k(\hat{x}, m_1m_2; \alpha_1\alpha_2; k, l)), \tag{3.4.29}$$

$1 \leq k, l \leq 2^{m_2(m_1-1)}$ is a $2^{m_2(m_1-1)} \times 2^{m_2(m_1-1)}$ matrix. Now

$$\mathbb{K}(\hat{x}, m_1m_2; \alpha_1\alpha_2) = S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2} \tag{3.4.30}$$

must be shown as follows.

Theorem 3.20. For any $m_1 \geq 2$, $m_2 \geq 2$ and $m_3 \geq 2$, let $S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}$ be given as in (3.4.18). Then,

$$X_{\hat{x};m_1,m_2,m_3+1;\alpha_1;\alpha_2} = S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}X_{\hat{x};m_1,m_2,m_3;\alpha_2}, \tag{3.4.31}$$

or equivalently, the recursive formula

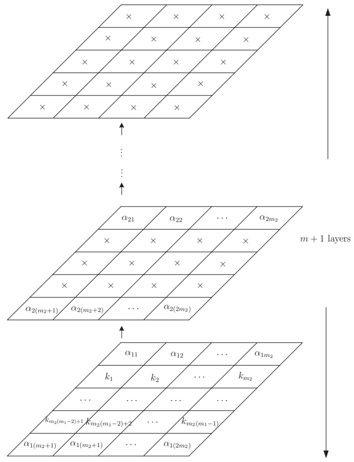
$$\begin{aligned} & A^{(k)}_{\hat{x};m_1,m_2,(m_3+1);\alpha_1} \\ = & \Big[\sum_{l=1}^{2^{m_2(m_1-1)}} (S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2})_{kl} A^{(l)}_{\hat{x};m_1;m_2,m_3;\alpha_2}\Big]_{2^{m_2}\times 2^{m_2}}, \end{aligned} \tag{3.4.32}$$

where $1 \leq \alpha_2 \leq 2^{2m_2}$. Moreover, for $m_3 = 1$,

$$A^{(k)}_{\hat{x};m_1,m_2,2;\alpha_1} = \Big[\sum_{l=1}^{2^{m_2(m_1-1)}} (S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2})_{kl}\Big]_{2^{m_2}\times 2^{m_2}}, \tag{3.4.33}$$

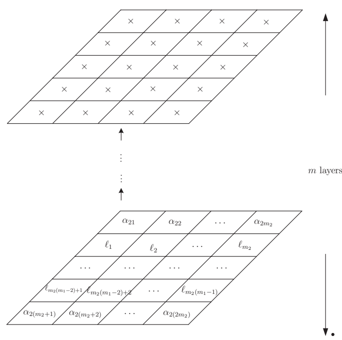
where $1 \leq \alpha_2 \leq 2^{2m_2}$ for any $1 \leq k \leq 2^{m_2(m_1-1)}$ and $\alpha_1 \in \{1, 2, \ldots, 2^{2m_2}\}$.

Proof. From (3.4.22), we can represent $A_{\hat{x};m_1,m_2,(m_3+1);\alpha_1 l\alpha_2}$ as the following patterns



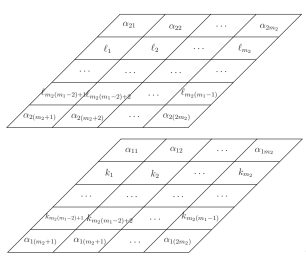
(3.4.34)

and $A_{\hat{x};m_1,m_2,m_3;\alpha_2^{(\ell)}}$ as the following patterns



(3.4.35)

By Definition 3.17, we get $S^{(r)}_{\hat{x};m_3;m_1m_2;\alpha_1;\alpha_2}$ represent the following pattern



(3.4.36)

Therefore, (3.4.32) follows from (3.4.34), (3.4.35) and (3.4.36). And by (3.4.28), (3.4.31) follows.

Next, (3.4.33) follows easily from (3.4.34) and (3.4.36). $\square$

For any positive integer $p \geq 2$, applying Theorem 3.20 p times permits the elementary patterns of $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times(m_3+p)}$ to be expressed as the product of a sequence of $S_{\hat{x};m_3;m_1m_2;\alpha_i\alpha_{i+1}}$ and the elementary patterns in $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times m_3}$. The elementary pattern in $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times(m_3+p)}$ is first studied. For any $p\geq 2$ and $1\leq q\leq p-1$, define

$$(3.4.37)\qquad \begin{aligned} & A^{(k)}_{\hat{x};(m_1+1),m_2,m_3+p;\alpha_1;\alpha_2;\ldots;\alpha_q} \\ = \ & [A^{(k)}_{\hat{x};(m_1+1),m_2,m_3+p;\alpha_1;\alpha_2;\ldots;\alpha_q;\alpha_{q+1}}]_{2^{m_2}\times 2^{m_2}}, \end{aligned}$$

where $1\leq\alpha_{q+1}\leq 2^{2m_2}$. Then $A^{(k)}_{\hat{x};m_1,m_2,m_3+p;\alpha_1;\alpha_2;\ldots;\alpha_p}$ could be represented as

$$(3.4.38)\qquad \sum_{l_1=1}^{2^{m_2(m_1-1)}}\sum_{l_2=1}^{2^{m_2(m_1-1)}}\cdots\sum_{l_p=1}^{2^{m_2(m_1-1)}}(\prod_{i=1}^{p}K(m_3;\alpha_{i-1},\alpha_i;l_{i-1},l_i))A^{(l_p)}_{\hat{x};m_1;m_2;m_3;\alpha_p}$$

where $\alpha_0=\alpha$ and $l_0=k$ can be easily verified.

Therefore, for any $p\geq 1$ a generalization for (3.4.24) can be found for $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times(m_3+p)}$ as a $(2^m_2)^{p+1}\times(2^m_2)^{p+1}$ matrix

$$(3.4.39)\qquad \mathbb{A}^{m_1}_{\hat{x};2\times m_2\times(m_3+p)}=[A_{\hat{x};m_1,m_2,(m_3+p);\alpha_1;\alpha_2;\ldots;\alpha_p}],$$

where

$$(3.4.40)\quad A^{(k)}_{\hat{x};m_1,m_2,(m_3+p);\alpha_1;\alpha_2;\ldots;\alpha_p}=\sum_{k=1}^{2^{2m_2}}A^{(k)}_{\hat{x};(m_1+1),m_2,m_3;\alpha_1;\alpha_2;\ldots;\alpha_p}.$$

In particular, if $\alpha_1,\alpha_2,\ldots,\alpha_p\in\{2^{m_2}(i-1)+i|1\leq i\leq 2^{m_2}\}$ then $A_{\hat{x};m_1,m_2,(m_3+p);\alpha_1;\alpha_2;\ldots;\alpha_p}$ lies on the diagonal of $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times(m_3+p)}$ in (3.4.41). Now, define

$$X_{\hat{x};m_1+1,m_2,m_3+p;\alpha_1;\alpha_2;\ldots;\alpha_p}=(A^{(k)}_{m_1+1,m_2,m_3+p;\alpha_1;\alpha_2;\ldots;\alpha_p})^t.$$

Therefore, Theorem 3.20 can be generalized to the following Theorem.

Theorem 3.21. For any $m_1\geq 2$, $m_2\geq 2$, $m_3\geq 2$ and $p\geq 1$, $X_{\hat{x};m_1,m_2,m_3+p;\alpha_1;\alpha_2;\ldots;\alpha_p}$ could be represented as

$$(3.4.41)\qquad S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3}\cdots S_{\hat{x};m_3;m_1m_2;\alpha_{p-1}\alpha_p}X_{\hat{x};m_1,m_2,m_3;\alpha_p}$$

where $1\leq\alpha_i\leq 2^{2m_2}$ and $1\leq i\leq p$.

Proof. From (3.4.38), (3.4.28) and (3.4.32),

$$
\begin{aligned}
& A_{\hat{x};m_1,m_2,m_3+p;\alpha_1;\alpha_2;\ldots;\alpha_p} \\
= & \sum_{\ell_1=1}^{2^{m_2(m_1-1)}} \sum_{\ell_2=1}^{2^{m_2(m_1-1)}} \cdots \sum_{\ell_p=1}^{2^{m_2(m_1-1)}} \Big(\prod_{i=1}^{p} K(\hat{x};m_3;\alpha_{i-1},\alpha_i;\ell_{i-1},\ell_i)\Big) A^{(\ell_p)}_{\hat{x};m_1,m_2,m_3;\alpha_p} \\
= & \sum_{\ell_1=1}^{2^{m_2(m_1-1)}} \sum_{\ell_2=1}^{2^{m_2(m_1-1)}} \cdots \sum_{\ell_p=1}^{2^{m_2(m_1-1)}} \Big(\prod_{i=1}^{p} (S_{\hat{x};m_3;m_1m_2;\alpha_{i-1}\alpha_{i-2}})_{\ell_{i-1}\ell_i}\Big) A^{(\ell_p)}_{\hat{x};m_1,m_2,m_3;\alpha_p} \\
= & \sum_{\ell_1=1}^{2^{m_2(m_1-1)}} \sum_{\ell_2=1}^{2^{m_2(m_1-1)}} \cdots \sum_{\ell_p=1}^{2^{m_2(m_1-1)}} \Big((S_{\hat{x};m_3;m_1m_2;\alpha_0\alpha_1})_{\ell_0\ell_1} (S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2})_{\ell_1\ell_2} \\
& \cdots (S_{\hat{x};m_3;m_1m_2;\alpha_{p-1}\alpha_p})_{\ell_{p-1}\ell_p}\Big) A^{(\ell_p)}_{\hat{x};m_1,m_2,m_3;\alpha_p} \\
= & \sum_{\ell_p=1}^{2^{m_2(m_1-1)}} (S_{\hat{x};m_3;m_1m_2;\alpha_0\alpha_1} S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2} \cdots S_{\hat{x};m_3;m_1m_2;\alpha_{p-1}\alpha_p})_{k\ell_p} A^{(\ell_p)}_{\hat{x};m_1,m_2,m_3;\alpha_p}.
\end{aligned}
$$

The proof is complete. ☐

3.4.2 Lower Bound of Entropy

In this subsection, the connecting operator $\mathbb{C}_{\hat{x};m_3;m_1m_2}$ is employed to estimate the lower bound of entropy and in particular, to verify the positivity of entropy.

Definition 3.22. Let $X = (X_1, \cdots, X_M)^t$, where X_k are $N \times N$ matrices. Define the summation of X_k by

$$|X| = \sum_{k=1}^{N} X_k. \tag{3.4.42}$$

If $\mathbb{M} = [M_{ij}]$ is a $M \times M$ matrix, then

$$|\mathbb{M}X| = \sum_{i=1}^{M} \sum_{j=1}^{M} M_{ij} X_j \tag{3.4.43}$$

Note that, (3.4.42) implies

$$|X_{\hat{x};m_1,m_2,m_3;\alpha}| = \sum_{k=1}^{2^{(m_1-1)m_2}} A^{(k)}_{\hat{x};m_1,m_2,m_3;\alpha} = A_{\hat{x};m_1,m_2,m_3;\alpha}. \tag{3.4.44}$$

As usual, the set of all matrices with the same order can be partially ordered.

Definition 3.23. Let $\mathbb{M} = [M_{ij}]$ and $\mathbb{N} = [N_{ij}]$ be two $M \times M$ matrices, $\mathbb{M} \geq \mathbb{N}$ if $M_{ij} \geq N_{ij}$ for all $1 \leq i, j \leq M$.

Notably, if $\mathbb{A}_2 \geq \mathbb{A}_2'$ then $\mathbb{A}_n \geq \mathbb{A}_n'$ for all $n \geq 2$. Therefore, $h(\mathbb{A}_2) \geq h(\mathbb{A}_2')$. Hence, the spatial entropy as a function of $\mathbb{A}_2$ is monotonic with respect to the partial order $\geq$.

Definition 3.24. A $P+1$ multiple index

$$\alpha_p \equiv (\alpha_1\alpha_2 \cdots \alpha_p\alpha_{P+1}) \tag{3.4.45}$$

is called a periodic cycle if

$$\alpha_{P+1} = \alpha_1. \tag{3.4.46}$$

It is called diagonal cycle if (3.4.46) holds and

$$\alpha_i \in \{2^{m_2(i-1)+i|1\leq i\leq 2^{2m_2}}\} \tag{3.4.47}$$

for each $1 \leq i \leq P+1$. For a diagonal cycle (3.4.45)

$$\bar{\alpha_P} = \alpha_1; \alpha_2; \cdots; \alpha_P \tag{3.4.48}$$

and

$$\bar{\alpha_P}^n = \bar{\alpha_P}; \bar{\alpha_P}; \cdots; \bar{\alpha_P}. \quad \text{(n-times)}$$

First, prove the following Lemma.

Lemma 3.25. Let $m_1 \geq 2$, $m_2 \geq 2$, $P \geq 1$, α_P be a diagonal cycle. Then, for any $m_3 \geq 1$,

$$\begin{aligned} &\rho(\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times(m_3P+2)}) \\ \geq\ &\rho(|(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3}\cdots S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_{P+1}})^{m_3}X_{\hat{x};m_1,m_2,2;\alpha_1}|). \end{aligned} \tag{3.4.49}$$

Proof. Since α_P is a periodic cycle, Theorem 3.21 implies

$$X_{\hat{x};m_1,m_2,m_3P+2;\bar{\alpha_P}^{m_1}} \tag{3.4.50}$$

$$= (S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3}\cdots S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_{P+1}})^nX_{\hat{x};m_1,m_2,2;\alpha_1}. \tag{3.4.51}$$

Furthermore α_P is diagonal and $|X_{\hat{x};m_1,m_2,m_3P+2;\bar{\alpha_P}^{m_1}}| = A_{\hat{x};m_1,m_2,m_3P+2;\bar{\alpha_P}^{m_1}}$ lies in the diagonal part of (3.4.41) with $m_3 + P = m_3P + 2$, therefore

$$\rho(\mathbb{A}^{m_1}_{\hat{x};m_1,m_2,m_3P+2}) \geq \rho(|X_{\hat{x};m_1,m_2,m_3P+2;\bar{\alpha_P}^{m_1}}|). \tag{3.4.52}$$

Therefore, (3.4.49) follows from (3.4.50) and (3.4.52). The proof is complete. $\square$

The following Lemma is valuable in studying maximum eigenvalue of (3.4.52).

Lemma 3.26. For any $m_1 \geq 2$, $m_2 \geq 2$, $1 \leq k \leq 2^{(m_1-1)m_2}$ and $\alpha_1 \in \{(i-1)2^{m_2}+i | 1 \leq i \leq 2^{m_2}\}$, if

$$tr(A^{(k)}_{\hat{x};m_1,m_2,2;\alpha}) = 0, \tag{3.4.53}$$

then for all $1 \leq \ell \leq 2^{(m_1-1)m_2}$,

$$(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2})_{k\ell} = 0, \tag{3.4.54}$$

for all $\alpha_2 \in \{(i-1)2^{m_2}+i | 1 \leq i \leq 2^{m_2}\}$, i.e., the k-th rows of matrices $S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}$ are zeros. Furthermore, for any diagonal cycle α_P, let $U = (u_1 u_2 \cdots u_{2^{m_2(m_1-1)}})$ be an eigenvector of $S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2} S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3} \cdots S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_1}$, if $u_k \neq 0$ for some $1 \leq k \leq 2^{(m_1-1)m_2}$, then $tr(A^{(k)}_{\hat{x};m_1,m_2,2;\alpha_1}) > 0$.

Proof. Since $A^{(k)}_{\hat{x};m_1,m_2,2;\alpha_1}$ can be expressed as (3.4.33). Therefore, $tr(A^{(k)}_{\hat{x};m_1,m_2,2;\alpha_1}) = 0$ if and only if (3.4.54) holds for all $1 \leq \ell \leq 2^{(m_1-1)m_2}$. The second part of the Lemma follows easily from the first part. The proof is complete. $\square$

By Lemma 3.25 and Lemma 3.26, the lower bound of entropy can be obtained as follows.

Theorem 3.27. Let $\alpha_1\alpha_2\cdots\alpha_P\alpha_1$ be a diagonal cycle. Then for any $m_1 \geq 2$, $m_2 \geq 2$,

$$\begin{aligned} & h(\mathbb{A}_{x;2\times2\times2}) \\ & \quad \geq \frac{1}{m_1m_2P} \log \rho(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2} S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3} \cdots S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_1}). \end{aligned} \tag{3.4.55}$$

In particular, if a diagonal cycle $\alpha_1\alpha_2\cdots\alpha_P\alpha_1$ exists and $m_1 \geq 2$, $m_2 \geq 2$ such that $\rho(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2} S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3} \cdots S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_1}) > 1$, then $h(\mathbb{A}_{x;2\times2\times2}) > 0$.

Proof. First, by the similar method in the proof of Lemma 2.10 and Lemma 2.11 and Theorem 2.12 in [5] we have

$$\begin{aligned} & \limsup_{m_3\to\infty} \frac{1}{m_3}(\log \rho(|(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2} S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3} \cdots S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_1})^n X_{\hat{x};m_1,m_2,2;\alpha_1}|)) \\ & \qquad = \log \rho(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2} S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3} \cdots S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_1}). \end{aligned} \tag{3.4.56}$$

Now, show that

$$\begin{aligned} h(\mathbb{A}_{x;2\times2\times2}) \geq \; & \frac{1}{m_1m_2P} \limsup_{m_3\to\infty} \frac{1}{m_3}(\log \rho(|(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2} S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3} \cdots \\ & S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_1})^n X_{\hat{x};m_1,m_2,2;\alpha_1}|)) \end{aligned}$$

Indeed, from (3.3.18) and (3.4.49),

$$\begin{aligned} h(\mathbb{A}_{x;2\times2\times2}) &= \lim_{m_2 m_3\to\infty} \frac{1}{(m_3P+2)m_2} \log \rho(\mathbb{A}_{\hat{x};2\times m_2\times(m_3P+2)}) \\ &= \lim_{m_2 m_3\to\infty} \frac{1}{m_1(m_3P+2)m_2} \log \rho(\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times(m_3P+2)}) \\ &\geq \frac{1}{m_1 m_2 P} \limsup_{m_3\to\infty} \frac{1}{m_3} (\log \rho(|(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2} S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3} \cdots \\ &\quad S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_1})^n X_{\hat{x};m_1,m_2,2;\alpha_1}|)). \end{aligned}$$

And by (3.4.56), the proof is complete. $\square$

Example 3.28. Consider

$$\mathbb{T}_{x;2\times2\times2} = \otimes(G\otimes E)^2.$$

Then, it is easy to check that

$$C_{\hat{x};m_3;22;11} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Therefore,

$$h(\mathbb{T}_{x;2\times2\times2}) \geq \frac{\log 2}{2}.$$

Moreover, in Proposition 3.15 it can be shown that $h(\mathbb{T}_{x;2\times2\times2}) = \log g$ where $g = \frac{1+\sqrt{5}}{2}$.

Reference

[1] J.C. Ban, K.P. Chien and S.S. Lin, *Spatial disorder of CNN-with asymmetric output funtion*, International J. of Bifurcation and Chaos **11**, (2001) 2085-2095.

[2] J.C. Ban, C.H. Hsu AND S.S.Lin, *Spatial disorder of cellular neural network-with biased term*, International J. of Bifurcation and Chaos **12**, (2002) 525-534.

[3] J.C. Ban, S.S. Lin and C.W. Shih, *Exact number of mosaic patterns in one-dimensional cellular neural networks*, International J. of Bifurcation and Chaos **11**, (2001) 1645-1653.

[4] J.C. Ban and S.S. Lin, *Patterns generation and transition matrices in multi-dimensional lattice models*, Discrete Contin. Dyn. Syst. **13**, (2005), no. 3, 637-658.

[5] J.C. Ban, S.S. Lin and Y.H. Lin, *Patterns generation and spatial entropy in two-dimensional lattice models*, Asian Journal of Mathematics, to appear.

[6] J.C. Ban, S.S. Lin and Y.H. Lin, *Primitivity of subshifts of finite type in two-dimensional lattice models*, preprint (2005).

[7] J.C. Ban, S.S. Lin and Y.H. Lin, *Patterns Generation and Spatial Entropy in Three-dimensional Lattice Models (II): Trace Operators*, preprint (2007).

[8] P.W. Bates and A. Chmaj, *A discrete convolution model for phase transitions*, Arch. Rat. Mech. Anal **150**, (1999) 281-305.

[9] P.W. Bates, K. Lu and B. Wang, *Attractors for lattice dynamical systems*, International J. of Bifurcation and Chaos **11**, (2001) 143-153.

[10] J. Bell, *Some threshold results for modes of myelinated nerves*, Math. Biosci. **54**, (1981) 181-190.

[11] J. Bell and C. Cosner, *Threshold behavior and propagation for nonlinear differential-difference systems motivated by modeling myelinated axons*, Quart. Appl. Math. **42**, (1984) 1-14.

[12] R. BELLMAN, *Introduction to matrix analysis*, Mc Graw-Hill, N.Y. (1970).

[13] R. J. Baxter, *Eight-vertex model in lattice statistics.*, Phys. Rev. Lett. **26**, (1971) 832-833.

[14] J.W. Cahn, *Theory of crystal growth and interface motion in crystalline materials*, Acta Metallurgica **8**, (1960) 554-562.

[15] S.N. Chow and J. Mallet-paret, *Pattern formation and spatial chaos in lattice dynamical systems II*, IEEE Trans. Circuits Systems **42**, (1995) 752-756.

[16] S.N. Chow, J. Mallet-paret and E. S. Van Vleck, *Dynamics of lattice differential equations*, International J. of Bifurcation and Chaos **6**, (1996) 1605-1621.

[17] S.N. Chow, J. Mallet-paret and E.S. Van Vleck, *Pattern formation and spatial chaos in spatially discrete evolution equations*, Random Comput. Dynam. **4**, (1996) 109-178.

[18] L.O. Chua, *CNN: A paradigm for complexity* (World Scientific Series on Nonlinear Science, Series A,31. World Scietific, Singapore, 1998)

[19] L.O. Chua, K.R. Crounse, M. Hasler and P. Thiran, *Pattern formation properties of autonomous cellular neural networks*, IEEE Trans. Circuits Systems **42**, (1995) 757-774.

[20] L.O. Chua and T. Roska, *The CNN paradigm*, IEEE Trans. Circuits Systems **40**, (1993) 147-156.

[21] L.O. Chua and L. Yang, *Cellular neural networks: Theory*, IEEE Trans. Circuits Systems **35**, (1988) 1257-1272.

[22] L.O. Chua and L. Yang, *Cellular neural networks: Applications*, IEEE Trans. Circuits Systems **35**, (1988) 1273-1290.

[23] G.B. Ermentrout, *Stable periodic solutions to discrete and continuum arrays of weakly coupled nonlinear oscillators*, SIAM J. Appl. Math. **52**, (1992) 1665-1687.

[24] G.B. Ermentrout and N. Kopell, *Inhibition-produced patterning in chains of coupled nonlinear oscillators*, SIAM J. Appl. Math. **54**, (1994) 478-507.

[25] G.B. Ermentrout, N. Kopell and T. L. Williams, *On chains of oscillators forced at one end*, SIAM J. Appl. Math. **51**, (1991) 1397-1417.

[26] T. Eveneux and J.P. Laplante, *Propagation failure in arrays of coupled bistable chemical reactors*, J. Phys. Chem. **96**, (1992) 4931-4934.

[27] W.J. Firth, *Optical memory and spatial chaos*, Phys. Rev. Lett. **61**, (1988) 329-332.

[28] F.R. Gantmacher, *The theory of matrices* (2 vols. Chelsea, New York, 1959)

[29] R.A. Horn and C.R. Johnson, *Matrix analysis* (Cambridge University Press, Cambridge, 1990)

[30] C.H. Hsu, J. Juang, S.S. Lin, and W.W. Lin, *Cellular neural networks: local patterns for general template*, International J. of Bifurcation and Chaos **10**, (2000) 1645-1659.

[31] J. Juang and S.S. Lin, *Cellular Neural Networks: Mosaic pattern and spatial chaos*, SIAM J. Appl. Math. **60**, (2000) 891-915.

[32] J. Juang, S.S. Lin, W.W. Lin and S.F. Shieh, *The spatial entropy of two-dimensional subshifts of finite type*, International J. of Bifurcation and Chaos **10**, (2000) 2845-2852.

[33] J.P. Keener, *Propagation and its failure in coupled systems of discrete excitable cells*, SIAM J. Appl. Math. **47**, (1987) 556-572.

[34] J.P. Keener, *The effects of discrete gap junction coupling on propagation in myocardium*, J. Theor. Biol. **148**, (1991) 49-82.

[35] A.L. Kimball, A. Varghese and R.L. Winslow, *Simulating cardiac sinus and atrial netwok dynamics on the connection machine*, Phys. D **64**, (1993) 281-298.

[36] E.H. Lieb, *Exact solution of the problem of the entropy of two-dimensional ice*, Phys. Rev. Lett. **18**, (1967) 692-694.

[37] E.H. Lieb, *Exact solution of the f model of an antiferroelectric*, Phys. Rev. Lett. **18**, (1967) 1046-1048.

[38] E.H. Lieb, *Exact Solution of the Two-Dimensional Slater KDP Model of a Ferroelectric*, Phys. Rev. Lett. **19**, (1967) 108-110.

[39] E.H. Lieb, *The Residual Entropy of Square Ice*, Phys. Rev. **162**, (1967) 162-172.

[40] E.H. Lieb, *Ice, ferro- and antiferroelectrics, in methods and problems in theoretical physics*, in honour of R.E. Peierls, proceedings of the 1967 Birmingham conference, (North-Holland, 1970) 21-28.

[41] S.S. Lin and T.S. Yang, *Spatial entropy of one-dimensional celluar neural network*, International J. of Bifurcation and Chaos **10**, (2000) 2129-2140.

[42] S.S. Lin and T.S. Yang, *On the spatial entropy and patterns of two-dimensional cellular neural networks*, International J. of Bifurcation and Chaos **12**, (2002) 115-128.

[43] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, (1995).

[44] A. Lindenmayer and P. Prusinkiewicz, *The algorithmic beauty of plants*, Springer-Verlag, New York, (1990).

[45] N.G. Markley and M.E. Paul, *Maximal measures and entropy for Z^ν subshifts of finite type.* Classical mechanics and dynamical systems, Medford, Mass., (1979) 135-157, Lecture Notes in Pure and Appl. Math., **70**, Dekker, New York, (1981).

[46] N.G. Markley and M.E. Paul, *Matrix subshifts for Z^ν symbolic dynamics*, Proc. London Math. Soc., (3) **43**, (1981) 251-272.

[47] L. Onsager, *Crystal statistics. I. A two-dimensional model with an order-disorder transition.*, Phys. Rev. **65**, (1944) 117-149.

[48] C.N. Yang and C.P. Yang, *One-dimensional chain of anisotropic spin-spin interactions. I. Proof of Bethe's hypothesis for ground state in a finite system.*, Phys. Rev. **150**, (1966a) 321-327.

[49] C.N. Yang and C.P. Yang, *One-dimensional chain of anisotropic spin-spin interactions. II. Properties of the ground-state energy per lattice site for an infinite system.*, Phys. Rev. **150**, (1966b) 327-339.

[50] C.N. Yang and C.P. Yang, *One-dimensional chain of anisotropic spin-spin interactions. III. Applications.*, Phys. Rev. **151**, (1966c) 258-264.